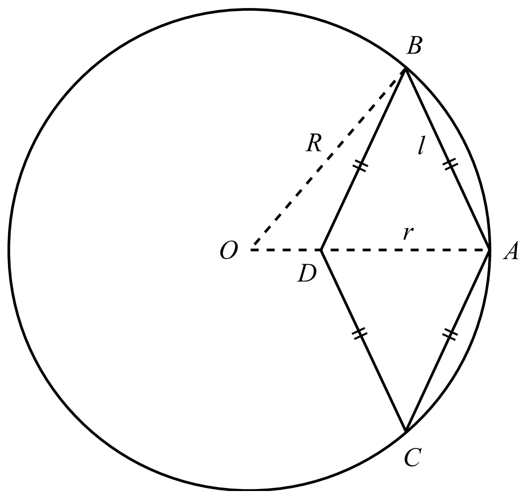
Solution

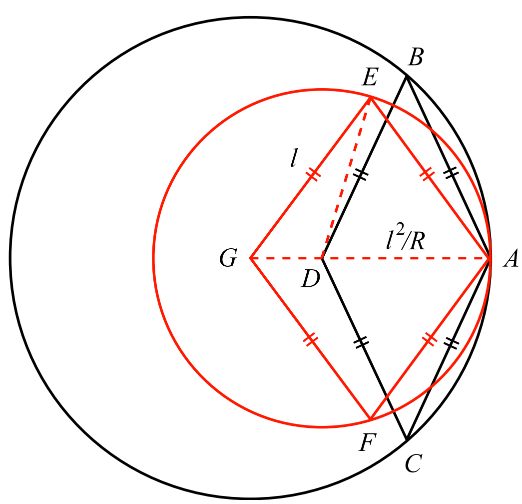
Week 52 (9/8/03)

Construct the center

Pick an arbitrary point A on the circle, as shown below. Construct points B and C on the circle, with $AB = AC = \ell$, where ℓ is arbitrary.[1] Construct point D with $DB = DC = \ell$. Let the distance DA be r. If O is the location (which we don't know yet) of the center of the circle, then triangles AOB and ABD are similar isosceles triangles (because they have $\angle BAD$ in common). Therefore, $OA/BA = BA/DA$, which gives $r = \ell^2/R$.



The above construction shows that if we are given a length ℓ and a circle of radius R, then we can construct the length ℓ^2/R. Therefore, we can produce the length R by simply repeating the above construction with the same length ℓ, but now with a circle of radius ℓ^2/R (which we just produced). In the following figure, we obtain $GA = \ell^2/(\ell^2/R) = R$. Hence, G is the center of the circle.



[1]However, this construction will not work if ℓ is too large or too small. We will determine these bounds below.

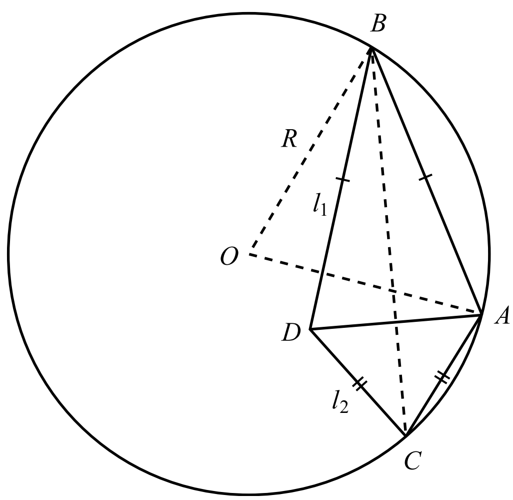
If you want to go through the similar-triangles argument, note that triangles ADE and AEG are similar isosceles triangles (because they have $\angle EAD$ in common). Therefore, $DA/EA = EA/GA$, which gives $GA = \ell^2/(\ell^2/R) = R$.

RESTRICTIONS: In order for this construction to work, it is necessary (and sufficient) for $R/2 < \ell < 2R$. The upper limit on ℓ comes from the requirement that a circle of radius ℓ (centered at A) intersects the given circle of radius R.[2] This gives $\ell < 2R$. The lower limit on ℓ comes from the requirement that a circle of radius ℓ (centered at A) intersects the circle of radius ℓ^2/R (centered at D). This gives $\ell < 2\ell^2/R$, which implies $R/2 < \ell$.

REMARK: The above solution can be extended to solve the following problem: Given three points, construct the circle passing through them.

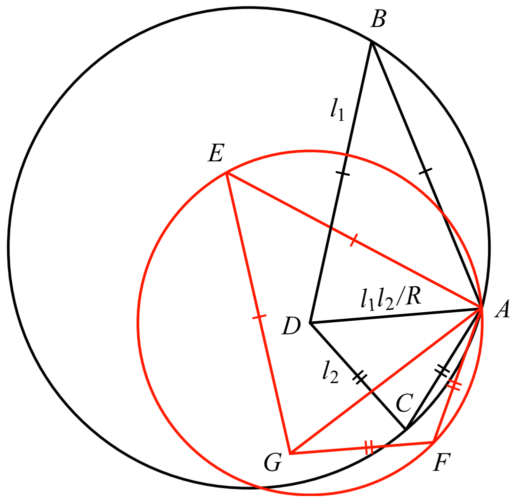
The solution proceeds along the lines of the above solution. In the figure below, construct point D with $DB = AB = \ell_1$ and $DC = AC = \ell_2$. Let O be the location (which we don't know yet) of the center of the desired circle. Then $\angle BOA = \widehat{BA}$. Also, $\angle DCA = 2(\angle BCA) = 2(\widehat{BA}/2) = \widehat{BA}$. Therefore, $\angle BOA = \angle DCA$, and so triangles BOA and DCA are similar isosceles triangles. Hence,

$$\frac{BO}{BA} = \frac{DC}{DA} \qquad \Longrightarrow \qquad \frac{R}{\ell_1} = \frac{\ell_2}{DA} \qquad \Longrightarrow \qquad DA = \frac{\ell_1 \ell_2}{R}\,. \tag{1}$$



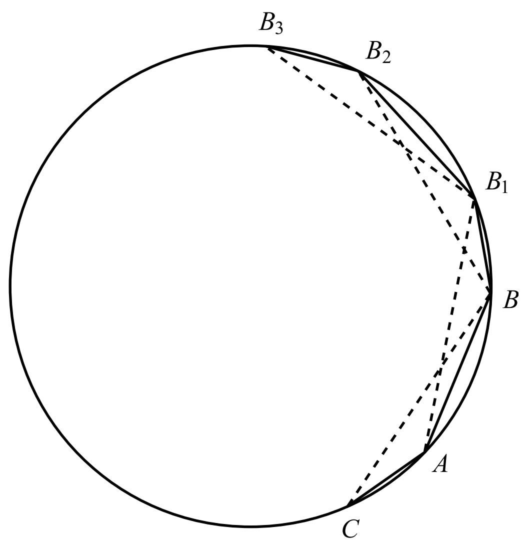
As in the above solution, we can apply this construction again, with the same lengths ℓ_1 and ℓ_2, but now with a circle of radius $\ell_1\ell_2/R$ (which we just produced). In the following figure, we obtain $GA = \ell_1\ell_2/(\ell_1\ell_2/R) = R$. And having found R, we can easily construct the center of the circle.

[2]The construction still works even if the intersection points are nearly diametrically opposite to A.



RESTRICTIONS: In order for this construction to work, we must be able to construct points E and F on the circle of radius $\ell_1\ell_2/R$ (centered at D). In order for these points to exist, the diameter of this circle must be larger than both ℓ_1 and ℓ_2. That is, $2\ell_1\ell_2/R > \max(\ell_1, \ell_2)$. This can be rewritten as $\min(\ell_1, \ell_2) > R/2$. This is the condition that must be satisfied. (There are no upper bounds on ℓ_1 and ℓ_2.)

What do we do if this lower bound is not satisfied? Simply construct more points on the circle until some three of them satisfy the constraint. For example, as shown below, construct point B_1 with $B_1B = CA$ and $B_1A = CB$. Then triangle B_1BA is congruent to triangle CAB, so point B_1 also lies on the circle. In a similar manner we can construct B_2 as shown, and then B_3, etc., to obtain an arbitrary number of points on the circle. After constructing many points, we will eventually be able to pick three of them that satisfy the constraint, $\min(\ell_1, \ell_2) > R/2$.



Of course, after constructing these new points on the circle, it is easy to pick three of them that have $\ell_1 = \ell_2$ (for example B, B_{2n}, and B_{4n}, in the notation in the figure). We can then use the easier, symmetrical solution in part (a) to find the center of the circle.