Solution

Week 53 (9/15/03)

Circles on the ice

By construction, the normal force from the ice exactly cancels all effects of the gravitational and centrifugal forces in the rotating frame of the earth (because the plumb bob hangs in the direction of the "effective gravity" force, which is the sum of the gravitational and centrifugal forces). We therefore need only concern ourselves with the Coriolis force. This force equals $\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$.

Let the angle down from the north pole be θ (we assume the circle is small enough so that θ is essentially constant throughout the motion). Then the component of the Coriolis force that points horizontally along the surface has magnitude $f = 2m\omega v \cos\theta$ and is perpendicular to the direction of motion. (The vertical component of the Coriolis force will simply modify the required normal force.) Because this force is perpendicular to the direction of motion, v does not change. Therefore, f is constant. But a constant force perpendicular to the motion of a particle produces a circular path. The radius of the circle is given by

$$2m\omega v \cos\theta = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{v}{2\omega\cos\theta}\,. \tag{1}$$

The frequency of the circular motion is

$$\omega' = \frac{v}{r} = 2\omega\cos\theta. \tag{2}$$

REMARKS: To get a rough idea of the size of the circle, you can show (using $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$) that $r \approx 10\,\text{km}$ when $v = 1\,\text{m/s}$ and $\theta = 45°$. Even the tiniest bit of friction will clearly make this effect essentially impossible to see.

For the $\theta \approx \pi/2$ (that is, near the equator), the component of the Coriolis force along the surface is negligible, so r becomes large, and ω' goes to 0.

For the $\theta \approx 0$ (that is, near the north pole), the Coriolis force essentially points along the surface. The above equations give $r \approx v/(2\omega)$, and $\omega' \approx 2\omega$. For the special case where the center of the circle is the north pole, this $\omega' \approx 2\omega$ result might seem incorrect, because you might want to say that the circular motion should be achieved by having the puck remain motionless in the inertial fame, while the earth rotates beneath it (thus making $\omega' = \omega$). The error in this reasoning is that the "level" earth is not spherical, due to the non-radial direction of the effective gravity (the combination of the gravitational and centrifugal forces). If the puck starts out motionless in the inertial frame, it will be drawn toward the north pole, due to the component of gravity in that direction. In order to not fall toward the pole, the puck needs to travel with frequency ω (relative to the inertial frame) in the direction opposite to the earth's rotation.[1] The puck therefore moves at frequency 2ω relative to the frame of the earth.

[1] In the rotating frame of the puck, the puck then feels the same centrifugal force that it would feel if it were at rest on the earth, spinning with it. It therefore happily stays at the same θ value on the "level" surface, just as a puck at rest on the earth does.