Solution

Week 54 (9/22/03)

Rolling the die

To get a feel for the problem, you can work out the answer for small values of N. For $N = 1$, the probability that the first player wins is 1. For $N = 2$, it is 3/4. And for $N = 3$, it is 19/27. A pattern in these numbers is more evident if we instead list the probabilities that the first player loses. These are 0, 1/4, and 8/27. (And if you work things out for $N = 4$, you'll obtain 81/256.) We therefore guess that the probability, P_L, that the first player *loses* is

$$P_L = \left(1 - \frac{1}{N}\right)^N. \qquad (1)$$

We'll prove this by proving the following more general claim. Eq. (1) is the special case of the claim with $r = 0$.

Claim: *Let L_r be the probability that a player loses, given that a roll of r has just occurred. Then*

$$L_r = \left(1 - \frac{1}{N}\right)^{N-r}. \qquad (2)$$

Proof: Assume that a roll of r has just occurred. To determine the probability, L_r, that the player who goes next loses, let's consider the probability, $1 - L_r$, that she wins. In order to win, she must roll a number, a, greater than r (each of which occurs with probability $1/N$); and her opponent must then lose, given that he has to beat a roll of a (this occurs with probability L_a). So the probability of winning, given that one must beat a roll of r, is

$$1 - L_r = \frac{1}{N}(L_{r+1} + L_{r+2} + \cdots + L_N). \qquad (3)$$

If we write down the analogous equation using $r - 1$ instead of r,

$$1 - L_{r-1} = \frac{1}{N}(L_r + L_{r+1} + \cdots + L_N), \qquad (4)$$

and then subtract eq. (4) from eq. (3), we obtain

$$L_{r-1} = \left(1 - \frac{1}{N}\right)L_r, \qquad (5)$$

for all r from 1 to N. Using $L_N = 1$, we find that

$$L_r = \left(1 - \frac{1}{N}\right)^{N-r} \qquad (0 \le r \le N). \ \blacksquare \qquad (6)$$

We may consider the first player to start out with a roll of $r = 0$ having just occurred. Therefore, the probability that the first player wins is $1 - (1 - 1/N)^N$. For large N, this probability approaches $1 - 1/e \approx 63.2\%$. For a standard die with $N = 6$, it takes the value $1 - (5/6)^6 \approx 66.5\%$.

Note that the probability that the first player wins can be written as

$$1-\left(1-\frac{1}{N}\right)^{N}=\frac{1}{N}\left(\left(1-\frac{1}{N}\right)^{N-1}+\left(1-\frac{1}{N}\right)^{N-2}+\cdots+\left(1-\frac{1}{N}\right)^{1}+1\right). \quad (7)$$

The right-hand side shows explicitly the probabilities of winning, depending on what the first roll is. For example, the first term on the right-hand side is the probability, $1/N$, that the first player rolls a 1, times the probability, $(1-1/N)^{N-1}$, that the second player loses given that he must beat a 1.