Solution

Week 54  (9/22/03)

Rolling the die

To get a feel for the problem, you can work out the answer for small values of \(N\). For \(N = 1\), the probability that the first player wins is 1. For \(N = 2\), it is \(3/4\). And for \(N = 3\), it is \(19/27\). A pattern in these numbers is more evident if we instead list the probabilities that the first player loses. These are 0, \(1/4\), and \(8/27\). (And if you work things out for \(N = 4\), you’ll obtain \(81/256\).) We therefore guess that the probability, \(P_L\), that the first player loses is

\[
P_L = \left(1 - \frac{1}{N}\right)^N.
\]

We’ll prove this by proving the following more general claim. Eq. (1) is the special case of the claim with \(r = 0\).

**Claim:** Let \(L_r\) be the probability that a player loses, given that a roll of \(r\) has just occurred. Then

\[
L_r = \left(1 - \frac{1}{N}\right)^{N-r}.
\]

**Proof:** Assume that a roll of \(r\) has just occurred. To determine the probability, \(L_r\), that the player who goes next loses, let’s consider the probability, \(1 - L_r\), that she wins. In order to win, she must roll a number, \(a\), greater than \(r\) (each of which occurs with probability \(1/N\)); and her opponent must then lose, given that he has to beat a roll of \(a\) (this occurs with probability \(L_a\)). So the probability of winning, given that one must beat a roll of \(r\), is

\[
1 - L_r = \frac{1}{N}(L_{r+1} + L_{r+2} + \cdots + L_N).
\]

If we write down the analogous equation using \(r - 1\) instead of \(r\),

\[
1 - L_{r-1} = \frac{1}{N}(L_r + L_{r+1} + \cdots + L_N),
\]

and then subtract eq. (4) from eq. (3), we obtain

\[
L_{r-1} = \left(1 - \frac{1}{N}\right)L_r,
\]

for all \(r\) from 1 to \(N\). Using \(L_N = 1\), we find that

\[
L_r = \left(1 - \frac{1}{N}\right)^{N-r} \quad (0 \leq r \leq N).
\]

We may consider the first player to start out with a roll of \(r = 0\) having just occurred. Therefore, the probability that the first player wins is \(1 - \left(1 - 1/N\right)^N\). For large \(N\), this probability approaches \(1 - 1/e \approx 63.2\%\). For a standard die with \(N = 6\), it takes the value \(1 - (5/6)^6 \approx 66.5\%\).
Note that the probability that the first player wins can be written as

$$1 - \left(1 - \frac{1}{N}\right)^N = \frac{1}{N} \left(\left(1 - \frac{1}{N}\right)^{N-1} + \left(1 - \frac{1}{N}\right)^{N-2} + \cdots + \left(1 - \frac{1}{N}\right)^1 + 1\right). \quad (7)$$

The right-hand side shows explicitly the probabilities of winning, depending on what the first roll is. For example, the first term on the right-hand side is the probability, $1/N$, that the first player rolls a 1, times the probability, $(1 - 1/N)^{N-1}$, that the second player loses given that he must beat a 1.