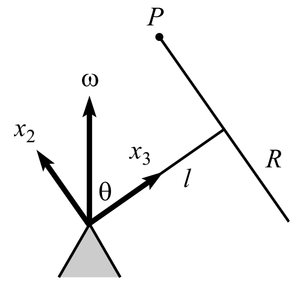
Solution

Week 55 (9/29/03)

Fixed highest point

For the desired motion, the important thing to note is that every point in the top moves in a fixed circle around the $\hat{\mathbf{z}}$-axis. Therefore, $\boldsymbol{\omega}$ points vertically. Hence, if Ω is the frequency of precession, we have $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}}$, as shown.



We'll need to use $\boldsymbol{\tau} = d\mathbf{L}/dt$ to solve this problem, so let's first calculate $\mathbf{L}$. With the pivot as the origin, the principle axes are shown above (and $\hat{\mathbf{x}}_1$ points into the page, but it won't come into play here). The principal moments are

$$I_3 = \frac{MR^2}{2}, \qquad \text{and} \qquad I \equiv I_1 = I_2 = M\ell^2 + \frac{MR^2}{4}, \tag{1}$$

where we have used the parallel-axis theorem to obtain the latter. The components of $\boldsymbol{\omega}$ along the principal axes are $\omega_3 = \Omega\cos\theta$, and $\omega_2 = \Omega\sin\theta$. Therefore, we have

$$\begin{aligned} \mathbf{L} &= I_3\omega_3\hat{\mathbf{x}}_3 + I_2\omega_2\hat{\mathbf{x}}_2 \\ &= I_3\Omega\cos\theta\hat{\mathbf{x}}_3 + I\Omega\sin\theta\hat{\mathbf{x}}_2, \end{aligned} \tag{2}$$

where we have kept things in terms of the moments, I_3 and I, to be general for now. The horizontal component of $\mathbf{L}$ is

$$\begin{aligned} L_\perp &= L_3\sin\theta - L_2\cos\theta \\ &= (I_3\Omega\cos\theta)\sin\theta - (I\Omega\sin\theta)\cos\theta \\ &= (I_3 - I)\Omega\cos\theta\sin\theta, \end{aligned} \tag{3}$$

with rightward taken to be positive. This horizontal component spins around in a circle with frequency Ω. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_\perp\Omega = \Omega^2\sin\theta\cos\theta(I_3 - I), \tag{4}$$

and it is directed into the page (or out of the page, if this quantity is negative). $d\mathbf{L}/dt$ must equal the torque, which has magnitude $|\boldsymbol{\tau}| = Mg\ell\sin\theta$ and is directed into the page. Therefore,

$$\Omega = \sqrt{\frac{Mg\ell}{(I_3 - I)\cos\theta}}\,. \tag{5}$$

We see that for a general symmetric top, the desired precessional motion (where the same "side" always points up) is possible only if

$$I_3 > I. \tag{6}$$

Note that this condition is independent of θ. For the problem at hand, I_3 and I are given in eq. (1), so we find

$$\Omega = \sqrt{\frac{4g\ell}{(R^2 - 4\ell^2)\cos\theta}}\,. \tag{7}$$

The necessary condition for the desired motion to exist is therefore

$$R > 2\ell. \tag{8}$$

REMARKS:

1. Given that the desired motion does indeed exist, it is intuitively clear that Ω should become very large as $\theta \to \pi/2$. But it is by no means intuitively clear (at least to me) that such motion should exist at all for angles near $\pi/2$.
2. Ω approaches a non-zero constant as $\theta \to 0$, which isn't entirely obvious.
3. If both R and ℓ are scaled up by the same factor, we see that Ω decreases. This also follows from dimensional analysis.
4. The condition $I_3 > I$ can be understood in the following way. If $I_3 = I$, then $\mathbf{L} \propto \boldsymbol{\omega}$, and so $\mathbf{L}$ points vertically along $\boldsymbol{\omega}$. If $I_3 > I$, then $\mathbf{L}$ points somewhere to the right of the $\hat{\mathbf{z}}$-axis (at the instant shown in the figure above). This means that the tip of $\mathbf{L}$ is moving into the page, along with the top. This is what we need, because $\boldsymbol{\tau}$ points into the page. If, however, $I_3 < I$, then $\mathbf{L}$ points somewhere to the left of the $\hat{\mathbf{z}}$-axis, so $d\mathbf{L}/dt$ points out of the page, and hence cannot be equal to $\boldsymbol{\tau}$.