Solution
Week 55  (9/29/03)

Fixed highest point

For the desired motion, the important thing to note is that every point in the top moves in a fixed circle around the $\hat{z}$-axis. Therefore, $\omega$ points vertically. Hence, if $\Omega$ is the frequency of precession, we have $\omega = \Omega \hat{z}$, as shown.

We’ll need to use $\tau = dL/dt$ to solve this problem, so let’s first calculate $L$. With the pivot as the origin, the principle axes are shown above (and $\hat{x}_1$ points into the page, but it won’t come into play here). The principal moments are

$$I_3 = \frac{MR^2}{2}, \quad \text{and} \quad I \equiv I_1 = I_2 = M\ell^2 + \frac{MR^2}{4}, \quad (1)$$

where we have used the parallel-axis theorem to obtain the latter. The components of $\omega$ along the principal axes are $\omega_3 = \Omega \cos \theta$, and $\omega_2 = \Omega \sin \theta$. Therefore, we have

$$L = I_3 \omega_3 \hat{x}_3 + I_2 \omega_2 \hat{x}_2 = I_3 \Omega \cos \theta \hat{x}_3 + I \Omega \sin \theta \hat{x}_2, \quad (2)$$

where we have kept things in terms of the moments, $I_3$ and $I$, to be general for now. The horizontal component of $L$ is

$$L_\perp = L_3 \sin \theta - L_2 \cos \theta = (I_3 \Omega \cos \theta) \sin \theta - (I \Omega \sin \theta) \cos \theta = (I_3 - I) \Omega \cos \theta \sin \theta, \quad (3)$$

with rightward taken to be positive. This horizontal component spins around in a circle with frequency $\Omega$. Therefore, $dL/dt$ has magnitude

$$\left| \frac{dL}{dt} \right| = L_\perp \Omega = \Omega^2 \sin \theta \cos \theta (I_3 - I), \quad (4)$$

and it is directed into the page (or out of the page, if this quantity is negative). $dL/dt$ must equal the torque, which has magnitude $|\tau| = Mg\ell \sin \theta$ and is directed into the page. Therefore,

$$\Omega = \sqrt{\frac{Mg\ell}{(I_3 - I) \cos \theta}}, \quad (5)$$
We see that for a general symmetric top, the desired precessional motion (where the same “side” always points up) is possible only if

\[ I_3 > I. \]  

(6)

Note that this condition is independent of \( \theta \). For the problem at hand, \( I_3 \) and \( I \) are given in eq. (1), so we find

\[ \Omega = \sqrt{\frac{4g\ell}{(R^2 - 4\ell^2)\cos \theta}}. \]  

(7)

The necessary condition for the desired motion to exist is therefore

\[ R > 2\ell. \]  

(8)

Remarks:

1. Given that the desired motion does indeed exist, it is intuitively clear that \( \Omega \) should become very large as \( \theta \to \pi/2 \). But it is by no means intuitively clear (at least to me) that such motion should exist at all for angles near \( \pi/2 \).

2. \( \Omega \) approaches a non-zero constant as \( \theta \to 0 \), which isn’t entirely obvious.

3. If both \( R \) and \( \ell \) are scaled up by the same factor, we see that \( \Omega \) decreases. This also follows from dimensional analysis.

4. The condition \( I_3 > I \) can be understood in the following way. If \( I_3 = I \), then \( \mathbf{L} \propto \omega \), and so \( \mathbf{L} \) points vertically along \( \omega \). If \( I_3 > I \), then \( \mathbf{L} \) points somewhere to the right of the \( \hat{z} \)-axis (at the instant shown in the figure above). This means that the tip of \( \mathbf{L} \) is moving into the page, along with the top. This is what we need, because \( \boldsymbol{\tau} \) points into the page. If, however, \( I_3 < I \), then \( \mathbf{L} \) points somewhere to the left of the \( \hat{z} \)-axis, so \( d\mathbf{L}/dt \) points out of the page, and hence cannot be equal to \( \boldsymbol{\tau} \).