Solution

Week 58 (10/20/03)

Coins and Gaussians

There are $\binom{2N}{N+x}$ ways to obtain $N+x$ heads in $2N$ flips. Therefore, the probability of obtaining $N+x$ heads is

$$P(x) = \frac{1}{2^{2N}}\binom{2N}{N+x}. \tag{1}$$

Our goal is to find an approximate expression for $\binom{2N}{N+x}$ when N is large. Using Stirling's formula,[1] $N! \approx N^N e^{-N}\sqrt{2\pi N}$, we have

$$\begin{aligned}
\binom{2N}{N+x} &= \frac{(2N)!}{(N+x)!(N-x)!} \\
&\approx \frac{(2N)^{2N}\sqrt{2N}}{(N+x)^{N+x}(N-x)^{N-x}\sqrt{2\pi}\sqrt{N^2-x^2}} \\
&= \frac{2^{2N}\sqrt{N}}{\left(1+\frac{x}{N}\right)^{N+x}\left(1-\frac{x}{N}\right)^{N+x}\sqrt{\pi}\sqrt{N^2-x^2}}\,.
\end{aligned} \tag{2}$$

Now,

$$\begin{aligned}
\left(1+\frac{x}{N}\right)^{N+x} &= \exp\Big((N+x)\ln\Big(1+\frac{x}{N}\Big)\Big) \\
&= \exp\Big((N+x)\Big(\frac{x}{N}-\frac{x^2}{2N^2}+\cdots\Big)\Big) \\
&\approx \exp\Big(x+\frac{x^2}{2N}\Big).
\end{aligned} \tag{3}$$

Likewise,

$$\left(1-\frac{x}{N}\right)^{N+x} \approx \exp\Big(-x+\frac{x^2}{2N}\Big). \tag{4}$$

Therefore, eq. (2) becomes

$$\binom{2N}{N+x} \approx \frac{2^{2N}e^{-x^2/N}}{\sqrt{\pi N}}\,, \tag{5}$$

where we have set $\sqrt{N^2-x^2}\approx N$, because the exponential factor shows that only x up to order $\sqrt{N}$ contribute significantly. Note that it is necessary to expand the log in eq. (3) to second order to obtain the correct result. Using eq. (5) in eq. (1) gives the desired result,

$$P(x) \approx \frac{e^{-x^2/N}}{\sqrt{\pi N}}\,. \tag{6}$$

Note that if we integrate this probability over x, we do indeed obtain 1, because $\int_{-\infty}^{\infty} e^{-x^2/N}\,dx = \sqrt{\pi N}$ (see the solution to Problem of the Week 56).

[1] From Problem of the Week 56.

REMARK: To find where eq. (6) is valid, we can expand the log factor in eq. (3) to fourth order in x. We then obtain

$$P(x) \approx \frac{e^{-x^2/N}}{\sqrt{\pi N}} e^{-x^4/6N^3}. \tag{7}$$

Therefore, when $x \sim N^{3/4}$, eq. (6) is not valid. However, when $x \sim N^{3/4}$, the $e^{-x^2/N}$ factor in $P(x)$ makes it negligibly small, so $P(x)$ is essentially zero in any case.