Solution

Week 59 (10/27/03)

Getting way ahead

The explanation of why the two clocks show different times in the ground frame is the following. The rocket becomes increasingly length contracted in the ground frame, which means that the front end isn't traveling as fast as the back end. Therefore, the time-dilation factor for the front clock isn't as large as that for the back clock. So the front clock loses less time relative to the ground, and hence ends up ahead of the back clock. Of course, it's not at all obvious that everything works out quantitatively, and that the front clock eventually ends up an arbitrarily large time ahead of the back clock. In fact, it's quite surprising that this is the case, since the above difference in speeds is rather small. But let's now show that the above explanation does indeed account for the difference in the clock readings.

Let the back of the rocket be located at position x. Then the front is located at position $x+L\sqrt{1-v^2}$, due to the length contraction. (We will set $c=1$ throughout this solution.) Taking the time derivatives of the two positions, we see that the speeds of the back and front are (with $v \equiv dx/dt$)[1]

$$v_b = v, \qquad v_f = v(1 - L\gamma\dot{v}), \qquad \text{where } \ \gamma \equiv 1/\sqrt{1-v^2}. \tag{1}$$

We must now find v in terms of the time, t, in the ground frame. The quickest way to do this is to use the fact that longitudinal forces are independent of the frame.[2] The force on, say, the astronaut, is $f = mg$ in the spaceship frame, so it must also be mg in the ground frame. Therefore, $F = dp/dt$ in the ground frame gives (using the fact that g is constant)

$$mg = \frac{d(m\gamma v)}{dt} \qquad \Longrightarrow \qquad gt = \gamma v \qquad \Longrightarrow \qquad v = \frac{gt}{\sqrt{1+(gt)^2}}\,. \tag{2}$$

REMARK: We can also find $v(t)$ by using the result of Problem of the Week 51, which says that in terms of the proper time, τ, of a uniformly-accelerated particle, its speed is given by $v(\tau) = \tanh(g\tau)$. This yields $\gamma = 1/\sqrt{1-v^2} = \cosh(g\tau)$. If we then integrate $dt = \gamma\, d\tau$, we obtain $gt = \sinh(g\tau)$. This gives

$$v(t) = \tanh(g\tau) = \frac{\sinh(g\tau)}{\cosh(g\tau)} = \frac{\sinh(g\tau)}{\sqrt{1+\sinh^2(g\tau)}} = \frac{gt}{\sqrt{1+(gt)^2}}\,. \quad \blacksquare \tag{3}$$

Having found v, we must now find the γ-factors associated with the speeds of the front and back of the rocket. The γ-factor associated with the speed of the back (namely v) is

$$\gamma_b = \frac{1}{\sqrt{1-v^2}} = \sqrt{1+(gt)^2}\,. \tag{4}$$

[1] Since these speeds are not equal, there is of course an ambiguity concerning which speed we should use in the length-contraction factor, $\sqrt{1-v^2}$. Equivalently, the rocket actually doesn't have one inertial frame that describes all of it. But you can show that any differences arising from this ambiguity are of higher order in gL/c^2 than we need to be concerned with.

[2] This takes a little effort to prove, but we'll just accept it here. If you want a self-contained method for finding v, we'll give another one in the remark below.

The γ-factor associated with the speed of the front, $v_f = v(1 - L\gamma\dot{v})$, is a little harder to obtain. We must first calculate $\dot{v}$. From eq. (2), we find $\dot{v} = g/(1+g^2t^2)^{3/2}$, which gives

$$v_f = v(1 - L\gamma\dot{v}) = \frac{gt}{\sqrt{1+(gt)^2}}\left(1 - \frac{gL}{1+g^2t^2}\right). \tag{5}$$

The γ-factor (or rather $1/\gamma$, which is what we'll be concerned with) associated with this speed can now be found as follows. In the first line below, we ignore the higher-order $(gL)^2$ term, because it is really $(gL/c^2)^2$, and we are assuming that gL/c^2 is small. And in obtaining the third line, we use the Taylor-series approximation $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$.

$$\begin{aligned} \frac{1}{\gamma_f} = \sqrt{1 - v_f^2} &\approx \sqrt{1 - \frac{g^2t^2}{1+g^2t^2}\left(1 - \frac{2gL}{1+g^2t^2}\right)} \\ &= \frac{1}{\sqrt{1+g^2t^2}}\sqrt{1 + \frac{2g^3t^2L}{1+g^2t^2}} \\ &\approx \frac{1}{\sqrt{1+g^2t^2}}\left(1 + \frac{g^3t^2L}{1+g^2t^2}\right). \end{aligned} \tag{6}$$

We can now calculate the time that each clock shows, at time t in the ground frame. The time on the back clock changes according to $dt_b = dt/\gamma_b$, so eq. (4) gives

$$t_b = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}}. \tag{7}$$

The integral[3] of $1/\sqrt{1+x^2}$ is $\sinh^{-1} x$. Letting $x \equiv gt$, this gives

$$gt_b = \sinh^{-1}(gt), \tag{8}$$

in agreement with the results in the above remark. The time on the front clock changes according to $dt_f = dt/\gamma_f$, so eq. (6) gives

$$t_f = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}} + \int_0^t \frac{g^3t^2L\,dt}{(1+g^2t^2)^{3/2}}. \tag{9}$$

The integral[4] of $x^2/(1+x^2)^{3/2}$ is $\sinh^{-1} x - x/\sqrt{1+x^2}$. Letting $x \equiv gt$, this gives

$$gt_f = \sinh^{-1}(gt) + (gL)\left(\sinh^{-1}(gt) - \frac{gt}{\sqrt{1+g^2t^2}}\right). \tag{10}$$

Using eqs. (8) and (2), we may rewrite this as

$$gt_f = gt_b(1 + gL) - gLv. \tag{11}$$

Dividing by g, and putting the c's back in to make the units correct, gives

$$t_f = t_b\left(1 + \frac{gL}{c^2}\right) - \frac{Lv}{c^2}, \tag{12}$$

as we wanted to show.

[3] To derive this, make the substitution $x \equiv \sinh\theta$.

[4] Again, to derive this, make the substitution $x \equiv \sinh\theta$.