

*Solution*

Week 6 (10/21/02)

**Flipping a coin**

- (a) There is a  $1/2$  chance that you win one dollar, a  $1/4$  chance that you win two dollars, a  $1/8$  chance that you win three dollars, etc. Therefore, the average value of your winnings is

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots \quad (1)$$

This may be written as

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right) + \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right) + \left(\frac{1}{8} + \frac{1}{16} + \dots\right) + \dots, \quad (2)$$

which equals

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) + \dots = 2. \quad (3)$$

So you expect to win an average of two dollars each time you play this game.

- (b) There is a  $1/2$  chance that you win one dollar, a  $1/4$  chance that you win two dollars, a  $1/8$  chance that you win four dollars, etc. So the average value of your winnings is now

$$\frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \dots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty. \quad (4)$$

You will quickly discover, however, that you will not win an infinite amount of money playing this game. We seem to have a paradox. The expectation value is infinite, but certainly no one is going to put up an infinite amount of money, or even a million dollars, for the opportunity to play this game once. What is the solution to this paradox?

The answer is that an expectation value is defined to be an average over an *infinite* number of trials (or the limit towards an infinite number), but you are simply not going to play an infinite number of games. In other words, the calculated expectation value doesn't agree with your experiment, because your experiment has nothing whatsoever to do with the precise definition of an expectation value. To be sure, if you did somehow play an infinite number of games, then you would indeed have an infinite average for your winnings. The whole paradox arises from trying to make "expectation value" mean something it doesn't.

This might not be a very satisfying explanation, so let us get a better feeling for the problem by looking at a situation where someone plays  $N = 2^n$  games. How much money would a "reasonable" person be willing to put up front for the opportunity to play these  $N$  games?

Well, in about  $2^{n-1}$  games he will win one dollar; in about  $2^{n-2}$  he will win two dollars; in about  $2^{n-3}$  games he will win four dollars; etc., until in about one

game he will win  $2^{n-1}$  dollars. In addition, there are the “fractional” numbers of games where he wins much larger quantities of money (for example, in half a game he will win  $2^n$  dollars, etc.), and this is indeed where the infinite expectation value comes from, in the calculation above. But let us forget about these for the moment, in order to just get a lower bound on what a reasonable person should put on the table. Adding up the above cases gives the total winnings as  $2^{n-1}(1) + 2^{n-2}(2) + 2^{n-3}(4) + \dots + 1(2^{n-1}) = 2^{n-1}n$ . The average value of these winnings in the  $N = 2^n$  games is therefore  $2^{n-1}n/2^n = n/2 = (\log_2 N)/2$ .

A reasonable person should therefore expect to win at least  $(\log_2 N)/2$  dollars per game. (By “expect”, we mean that if the player plays a very large number of sets of  $N$  games, and then takes an average over these sets, he will win at least  $2^{n-1}n$  dollars per set.) This clearly increases with  $N$ , and goes to infinity as  $N$  goes to infinity. It is nice to see that we can obtain this infinite limit without having to worry about what happens in the infinite number of “fractional” games. Remember, though, that this quantity,  $(\log_2 N)/2$ , has nothing to do with a true expectation value, which is only defined for  $N \rightarrow \infty$ .

Someone may still not be satisfied and want to ask, “But what if I play only  $N$  games? I will never ever play another game. How much money do I expect to win?” The proper answer is that the question has no meaning. It is not possible to define how much one expects to win, if one is not willing to take an average over a arbitrarily large number of trials.