Solution

Week 61 (11/10/03)

Falling rope

(a) **First Solution:** Let σ be the mass density of the rope. From conservation of energy, we know that the rope's final kinetic energy, which is $(\sigma L)v^2/2$, equals the loss in potential energy. This loss equals $(\sigma L)(L/2)g$, because the center of mass falls a distance $L/2$. Therefore,

$$v = \sqrt{gL}\,. \tag{1}$$

This is the same as the speed obtained by an object that falls a distance $L/2$. Note that if the initial piece hanging down through the hole is arbitrarily short, then the rope will take an arbitrarily long time to fall down. But the final speed will be always be (arbitrarily close to) $\sqrt{gL}$.

Second Solution: Let x be the length that hangs down through the hole. The gravitational force on this length, which is $(\sigma x)g$, is responsible for changing the momentum of the entire rope, which is $(\sigma L)\dot{x}$. Therefore, $F = dp/dt$ gives $(\sigma x)g = (\sigma L)\ddot{x}$, which is simply the $F = ma$ equation. Hence, $\ddot{x} = (g/L)x$, and the general solution to this equation is

$$x(t) = Ae^{t\sqrt{g/L}} + Be^{-t\sqrt{g/L}}. \tag{2}$$

Note that if ϵ is the initial value for x, then $A = B = \epsilon/2$ satisfies the initial conditions $x(0) = \epsilon$ and $\dot{x}(0) = 0$, in which case we may write $x(t) = \epsilon\cosh(t\sqrt{g/L})$. But we won't need this information in what follows.

Let T be the time for which $x(T) = L$. If ϵ is very small, then T will be very large. But for large t,[1] we may neglect the negative-exponent term in eq. (2). We then have

$$x \approx Ae^{t\sqrt{g/L}} \qquad \Longrightarrow \qquad \dot{x} \approx Ae^{t\sqrt{g/L}}\sqrt{g/L} \approx x\sqrt{g/L} \qquad \text{(for large } t\text{)}. \tag{3}$$

When $x = L$, we obtain

$$\dot{x}(T) = L\sqrt{g/L} = \sqrt{gL}\,, \tag{4}$$

in agreement with the first solution.

(b) Let σ be the mass density of the rope, and let x be the length that hangs down through the hole. The gravitational force on this length, which is $(\sigma x)g$, is responsible for changing the momentum of the rope. This momentum is $(\sigma x)\dot{x}$, because only the hanging part is moving. Therefore, $F = dp/dt$ gives

$$\sigma x g = \frac{d(\sigma x\dot{x})}{dt} \quad \Longrightarrow \quad xg = x\ddot{x} + \dot{x}^2. \tag{5}$$

[1]More precisely, for $t \gg \sqrt{L/g}$.

Note that $F = ma$ gives the wrong equation, because it neglects the fact that the moving mass, σx, is changing. It therefore misses the second term on the right-hand side of eq. (5). In short, the momentum of the rope increases because it is speeding up (which gives the $x\ddot{x}$ term) *and* because additional mass is continually being added to the moving part (which gives the $\dot{x}^2$ term, as you can show).

To solve eq. (5) for $x(t)$, note that g is the only parameter in the equation. Therefore, the solution for $x(t)$ can involve only g's and t's.[2] By dimensional analysis, $x(t)$ must then be of the form $x(t) = bgt^2$, where b is a numerical constant to be determined. Plugging this expression for $x(t)$ into eq. (5) and dividing by g^2t^2 gives $b = 2b^2 + 4b^2$. Therefore, $b = 1/6$, and our solution may be written as

$$x(t) = \frac{1}{2}\left(\frac{g}{3}\right)t^2. \tag{6}$$

This is the equation for something that accelerates downward with acceleration $g' = g/3$. The time the rope takes to fall a distance L is then given by $L = g't^2/2$, which yields $t = \sqrt{2L/g'}$. The final speed in thus

$$v = g't = \sqrt{2Lg'} = \sqrt{\frac{2gL}{3}}\,. \tag{7}$$

This is smaller than the $\sqrt{gL}$ result from part (a). We therefore see that although the total time for the scenario in part (a) is very large, the final speed in that case is in fact larger than that in the present scenario.

REMARKS: Using eq. (7), you can show that 1/3 of the available potential energy is lost to heat. This inevitable loss occurs during the abrupt motions that suddenly bring the atoms from zero to non-zero speed when they join the moving part of the rope. The use of conservation of energy is therefore *not* a valid way to solve this problem.

You can show that the speed in part (a)'s scenario is smaller than the speed in part (b)'s scenario for x less than $2L/3$, but larger for x greater than $2L/3$.

[2]The other dimensionful quantities in the problem, L and σ, do not appear in eq. (5), so they cannot appear in the solution. Also, the initial position and speed (which will in general appear in the solution for $x(t)$, because eq. (5) is a second-order differential equation) do not appear in this case, because they are equal to zero.