

Solution

Week 62 (11/17/03)

Leftover dental floss

Let (x, y) denote the occurrence where x segments have been cut off the left roll, and y segments have been cut off the right roll. In solving this problem, we'll need to first calculate the probability that the process ends at (N, n) , which leaves a length $(N - n)d$ on the right roll. For this to happen, the process must first get to $(N - 1, n)$, and then the left roll must be chosen for the last segment. The probability of reaching the point $(N - 1, n)$ is

$$P_{N-1,n} = \frac{1}{2^{N-1+n}} \binom{N-1+n}{n}, \quad (1)$$

because the binomial coefficient gives the number of different ways the $N - 1 + n$ choices of roll (each of which occurs with probability $1/2$) can end up at the point $(N - 1, n)$. The probability of then choosing the left roll for the next piece is $1/2$. Therefore, the probability of ending the process at (N, n) is

$$P_{N,n}^{\text{end}} = \frac{1}{2^{N+n}} \binom{N-1+n}{n}. \quad (2)$$

The leftover length in this case is $(N - n)d$, so we see that the average leftover length at the end of the process is

$$\begin{aligned} \ell &= 2d \sum_{n=0}^{N-1} (N - n) P_{N,n}^{\text{end}} \\ &= 2d \sum_{n=0}^{N-1} \frac{N - n}{2^{N+n}} \binom{N-1+n}{n}, \end{aligned} \quad (3)$$

where the factor of 2 out front comes from the fact that either roll may be the one that runs out.

In order to simplify this result, we will use the standard result that for large N , a binomial coefficient can be approximated by a Gaussian function. From Problem of the Week 58, we have, for large N and $x \ll N$,

$$\binom{2M}{M-x} \approx \frac{2^{2M}}{\sqrt{\pi M}} e^{-x^2/M}. \quad (4)$$

To make use of this, we'll first need to rewrite eq. (3) as

$$\begin{aligned} \ell &= 2d \sum_{n=0}^{N-1} \frac{N - n}{2^{N+n}} \frac{N}{N + n} \binom{N + n}{n} \\ &= 2d \sum_{z=1}^N \frac{z}{2^{2N-z}} \frac{N}{2N - z} \binom{2N - z}{N - z} \quad (\text{with } z \equiv N - n) \\ &= 2d \sum_{z=1}^N \frac{z}{2^{2N-z}} \frac{N}{2N - z} \binom{2(N - z/2)}{(N - z/2) - z/2}. \end{aligned} \quad (5)$$

Using eq. (4) to rewrite the binomial coefficient gives (with $M \equiv N - z/2$, and $x \equiv z/2$)

$$\begin{aligned}
\ell &\approx 2d \sum_{z=1}^N \frac{Nz}{2N-z} \frac{e^{-z^2/4(N-z/2)}}{\sqrt{\pi(N-z/2)}} \\
&\approx \frac{d}{\sqrt{\pi N}} \sum_{z=0}^N z e^{-z^2/(4N)} \\
&\approx \frac{d}{\sqrt{\pi N}} \int_0^\infty z e^{-z^2/(4N)} dz \\
&= 2d \sqrt{\frac{N}{\pi}}. \tag{6}
\end{aligned}$$

In obtaining the second line above, we have kept only the terms of leading order in N . The exponential factor guarantees that only z values up to order \sqrt{N} will contribute. Hence, z is negligible when added to N .

Without doing all the calculations, it's a good bet that the answer should go like $\sqrt{N}d$, but it takes some effort to show that the exact coefficient is $2/\sqrt{\pi}$.

In terms of the initial length of floss, $L \equiv Nd$, the average leftover amount can be written as $\ell \approx (2/\sqrt{\pi})\sqrt{Ld}$. So we see that ℓ is proportional to the geometric mean of L and d .