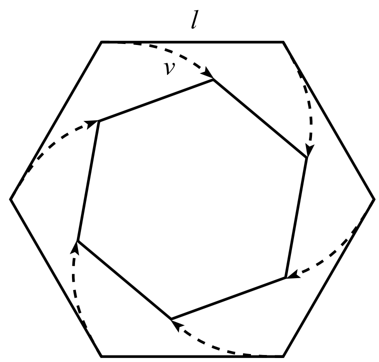
Solution

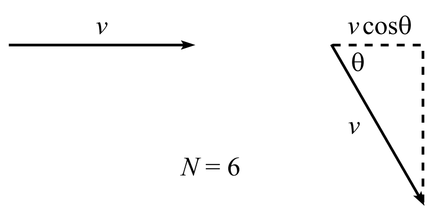
Week 64 (12/1/03)

Attracting bugs

First Solution: In all of these solutions, the key point to realize is that at any time, the bugs form the vertices of a regular N-gon, as shown below for $N = 6$. This is true because this is the only configuration that respects the symmetry of the N bugs. The N-gon will rotate and shrink until it becomes a point at the center.



The important quantity in this first solution is the relative speed of two adjacent bugs. This relative speed is constant, because the relative angle of the bugs' motions is always the same. If the bugs' speed is v, then we see from the figure below that the relative speed is $v_r = v(1 - \cos\theta)$, where $\theta = 2\pi/N$. This is the rate at which the separation between two adjacent bugs decreases.[1]



For example, if $N = 3$ we have $v_r = 3v/2$; if $N = 4$ we have $v_r = v$; and if $N = 6$ we have $v_r = v/2$. Note also that for $N = 2$ (which does not give not much of a polygon, being just a straight line) we have $v_r = 2v$, which is correct for two bugs walking directly toward each other. And if $N \to \infty$ we have $v_r \to 0$, which is correct for bugs walking around in a circle.

If two bugs start a distance ℓ apart, and if they always walk at a relative speed of $v(1 - \cos\theta)$, then the time it takes for them to meet is $t = \ell/(v(1 - \cos\theta))$. Therefore, since the bugs walk at speed v, they will each travel a total distance of

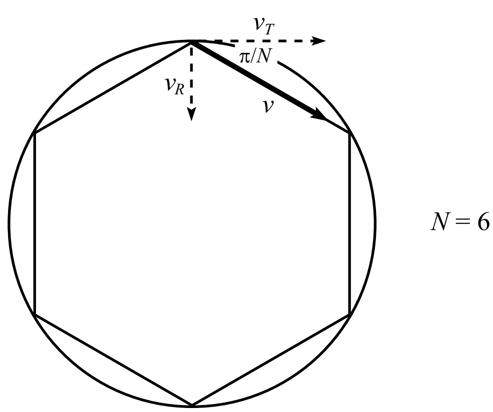
$$vt = \frac{\ell}{1 - \cos(2\pi/N)} \,. \tag{1}$$

[1] The transverse $v\sin\theta$ component of the front bug's velocity is irrelevant here, because it provides no first-order change in the distance between the bugs, for small increments in time.

Note that for a square, this distance equals the length of a side, ℓ. For large N, the approximation $\cos \approx 1 - \theta^2/2$ gives $vt \approx N^2\ell/(2\pi^2)$.

The bugs will spiral around an infinite number of times. This is true because the future path of the bugs at any time must simply be a scaled-down version of the future path at the start (because any point in time may be considered to be the start time, with a scaled-down version of the initial separation). This would not be possible if the bugs hit the center after spiraling around only a finite number of times. We will see in the third solution below that the bugs' distance from the center deceases by a factor $e^{-2\pi \tan(\pi/N)}$ after each revolution.[2]

Second Solution: In this solution, we will determine how quickly the bugs approach the center of the N-gon. A bug's velocity may be separated into radial and tangential components, v_R and v_T, as shown below. Because at any instant the bugs all lie on the vertices of a regular N-gon, they always walk at the same angle relative to a circular motion. Therefore, the magnitudes of v_R and v_T remain constant.



What is the radial component, v_R, in terms of v? The angle between a bug's motion and a circular motion is π/N, so we have

$$v_R = v \sin(\pi/N). \tag{2}$$

What is the radius, R_0, of the initial N-gon? A little geometry shows that

$$R_0 = \frac{\ell}{2\sin(\pi/N)}\,. \tag{3}$$

The time it takes a bug to reach the center is then $t = R_0/v_R = (\ell/v)/(2\sin^2(\pi/N))$. Therefore, each bug travels a total distance of

$$vt = \frac{\ell}{2\sin^2(\pi/N)}\,. \tag{4}$$

[2]Of course, bugs of non-zero size would hit each other before they reach the center. If the bugs happen to be very, very small, then they would eventually require arbitrarily large friction with the floor, in order to provide the centripetal acceleration needed to keep them in a spiral with a very small radius.

This agrees with the result of the first solution, due to the half-angle formula, $\sin^2(\theta/2) = (1-\cos\theta)/2$. The same reasoning used in the first solution shows that the bugs spiral around an infinite number of times.

Third Solution: In this solution, we will parametrize a bug's path, and then integrate the differential arclength. Let us find a bug's distance, $R(\phi)$, from the center, as a function of the angle ϕ through which it has travelled. The angle between a bug's motion and a circular motion is π/N. Therefore, the change in radius, dR, divided by the change in arclength along the circle, $R\,d\phi$, is $dR/(R\,d\phi) = -\tan(\pi/N)$. Separating variables and integrating gives

$$\begin{aligned} \int_{R_0}^{R} \frac{dR}{R} &= -\int_0^{\phi} \tan(\pi/N)\,d\phi \\ \Longrightarrow \quad \ln(R/R_0) &= -\phi\tan(\pi/N) \\ \Longrightarrow \quad R(\phi) &= R_0\,e^{-\phi\tan(\pi/N)}, \end{aligned} \tag{5}$$

where R_0 is the initial distance from the center, equal to $\ell/(2\sin(\pi/N))$. We now see, as stated in the first solution, that one revolution decreases R by the factor $e^{-2\pi\tan(\pi/N)}$, and that an infinite number of revolutions is required for R to become zero. Having found $R(\phi)$, we may find the total distance travelled by integrating the arclength:

$$\begin{aligned} \int \sqrt{(R\,d\phi)^2 + (dR)^2} &= \int_0^{\infty} \sqrt{R^2 + (dR/d\phi)^2}\,d\phi \\ &= \int_0^{\infty} \frac{R_0\,e^{-\phi\tan(\pi/N)}\,d\phi}{\cos(\pi/N)} \\ &= \frac{\ell}{2\sin^2(\pi/N)}\,. \end{aligned} \tag{6}$$

REMARK: In the first solution, we found that for large N the total distance travelled goes like $\ell N^2/(2\pi^2)$. This result can also be found in the following manner. For large N, a bug's motion can be approximated by a sequence of circles, C_n, with radii $R_n = R_0 e^{-n(2\pi)\tan(\pi/N)} \approx R_0 e^{-n(2\pi^2/N)}$. To leading order in N, the total distance travelled is therefore the sum of the geometric series,

$$\begin{aligned} \sum_{n=0}^{\infty} 2\pi R_n &\approx \sum_{n=0}^{\infty} 2\pi R_0 e^{-n(2\pi^2/N)} \\ &= \frac{2\pi R_0}{1-e^{-2\pi^2/N}} \\ &\approx \frac{2\pi R_0}{2\pi^2/N} \\ &\approx \frac{N^2\ell}{2\pi^2}\,, \end{aligned} \tag{7}$$

where we have used $R_0 = \ell/(2\sin(\pi/N)) \approx N\ell/(2\pi)$.