Solution

Week 65 (12/8/03)

Relativistic cart

Ground frame (your frame): In your frame, the force at your feet is responsible for changing the momentum of the cart-plus-sand-inside system. Let's label this system as "C".

To find the dp/dt of C, let's determine how fast the mass of C increases. We claim that the rate of mass increase is $\gamma\sigma$. This can be seen as follows. Assume that C has a mass M at a given time. A mass $\sigma\, dt$ falls into the cart during a time dt. The energy of the resulting C$'$ is $\gamma M + \sigma\, dt$ (we'll drop the c's here), while the momentum is still $\gamma M v$. Using $E^2 = p^2 + m^2$, we see that the resulting mass equals

$$M' = \sqrt{(\gamma M + \sigma\, dt)^2 - (\gamma M v)^2} \approx \sqrt{M^2 + 2\gamma M \sigma\, dt}, \tag{1}$$

where we have dropped the second-order dt^2 terms. Using the Taylor series $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$, we may approximate M' as

$$M' \approx M\sqrt{1 + \frac{2\gamma\sigma\, dt}{M}} \approx M\left(1 + \frac{\gamma\sigma\, dt}{M}\right) = M + \gamma\sigma\, dt. \tag{2}$$

Therefore, C's mass increases at a rate $\gamma\sigma$.[1] Intuitively, this rate of increase must certainly be greater than the nonrelativistic result of "σ", because heat is generated during the collision, and this heat shows up as mass in the final object.

Having found the rate at which the mass increases, we see that the rate at which the momentum increases is (using the fact that v is constant)

$$\frac{dp}{dt} = \gamma\left(\frac{dM}{dt}\right)v = \gamma(\gamma\sigma)v = \gamma^2\sigma v. \tag{3}$$

Since $F = dp/dt$, this is the force that you exert on the cart. Therefore, it is also the force that the ground exerts on your feet (because the net force on you is zero).

Cart frame: The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame; that is, at a rate σ/γ. The sand flies in at speed v, and then eventually comes at rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$. This is the force that your hand applies to the cart.

If this were the only change in momentum in the problem, then we would have a problem, because the force on your feet would be σv in the cart frame, whereas we found above that it is $\gamma^2\sigma v$ in the ground frame. This would contradict the fact that longitudinal forces are the same in different frames. What is the resolution of this apparent paradox?

[1] This result is easier to see if we work in the frame where C is at rest. In this frame, a mass $\sigma\, dt$ comes flying in with energy $\gamma\sigma\, dt$, and essentially all of this energy shows up as mass (heat) in the final object. That is, essentially none of it shows up as overall kinetic energy of the object, which is a general result for when a small object hits a stationary large object.

The resolution is that while you are pushing on the cart, *your mass is decreasing.* You are moving with speed v in the cart frame, and mass is continually being transferred from you (who are moving) to the cart (which is at rest). This is the missing change in momentum we need. Let's be quantitative about this.

Go back to the ground frame for a moment. We found above that the mass of C increases at rate $\gamma\sigma$ in the ground frame. Therefore, the energy of C increases at a rate $\gamma(\gamma\sigma)$ in the ground frame. The sand provides σ of this energy, so you must provide the remaining $(\gamma^2 - 1)\sigma$ part. Therefore, since you are losing energy at this rate, you must also be losing mass at this rate in the ground frame (because you are at rest there).

Now go back to the cart frame. Due to time dilation, you lose mass at a rate of only $(\gamma^2 - 1)\sigma/\gamma$. This mass goes from moving at speed v (that is, along with you), to speed zero (that is, at rest on the cart). Therefore, the rate of decrease in momentum of this mass is $\gamma((\gamma^2 - 1)\sigma/\gamma)v = (\gamma^2 - 1)\sigma v$.

Adding this result to the σv result due to the sand, we see that the total rate of decrease in momentum is $\gamma^2\sigma v$. This is therefore the force that the ground applies to your feet, in agreement with the calculation in the ground frame.