Solution

Week 66 (12/15/03)

Bowl of spaghetti

Assume that we have reached into the bowl and pulled out one end. Then there are $2N-1$ free ends left in the bowl. Therefore, there is a $1/(2N-1)$ chance that a loop is formed by choosing the other end of the noodle that we are holding. And there is a $(2N-2)/(2N-1)$ chance that a loop is not formed. In the former case, we end up with one loop and $N-1$ strands. In the latter case, we just end up with $N-1$ strands, because we have simply created a strand of twice the original length, and the length of a strand is irrelevant in this problem.

Therefore, after the first step, we see that no matter what happens, we end up with $N-1$ strands and, on average, $1/(2N-1)$ loops. We can now repeat this reasoning with $N-1$ strands. After the second step, we are guaranteed to be left with $N-2$ strands and, on average, another $1/(2N-3)$ loops. This process continues until we are left with one strand, whereupon the final Nth step leaves us with zero strands, and we (definitely) gain one more loop.

Adding up the average number of loops gained at each stage, we obtain an average total number of loops equal to

$$n = \frac{1}{2N-1} + \frac{1}{2N-3} + \cdots + \frac{1}{3} + 1. \tag{1}$$

This grows very slowly with N. It turns out that we need $N = 8$ noodles in order to expect at least two loops. If we use the ordered pair (n, N) to signify that N noodles are needed in order to expect n loops, we can numerically show that the first few integer-n ordered pairs are: $(1,1)$, $(2,8)$, $(3,57)$, $(4,419)$, and $(5,3092)$.

For large N, we can say that the average number of loops given in eq. (1) is roughly equal to $1/2$ times the sum of the reciprocals up to $1/N$. So it approximately equals $(\ln N)/2$. To get a better approximation, let S_N denote the sum of the integer reciprocals up to $1/N$. Then we have (using $S_N \approx \ln N + \gamma$, where $\gamma \approx 0.577$ is Euler's constant)

$$\begin{aligned}
n + \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2N-2} + \frac{1}{2N}\right) &= S_{2N} \\
\Longrightarrow \quad n + \frac{1}{2}S_N &= S_{2N} \\
\Longrightarrow \quad n + \frac{1}{2}(\ln N + \gamma) &\approx \ln(2N) + \gamma \\
\Longrightarrow \quad n &\approx \frac{1}{2}(\ln N + \ln 4 + \gamma) \\
\Longrightarrow \quad N &\approx \frac{e^{2n-\gamma}}{4}\,.
\end{aligned} \tag{2}$$

You can show that this relation between n and N agrees with the above numerical results, except for the $n = 1$ case. You will need to use the more precise value of $\gamma \approx 0.5772$ for the $n = 5$ case.