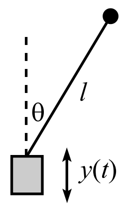
Solution

Week 67 (12/22/03)

Inverted pendulum

Let θ be defined as shown below. We'll use the Lagrangian method to determine the equation of motion for θ.



With $y(t) = A\cos(\omega t)$, the position of the mass m is given by

$$(X, Y) = (\ell\sin\theta, y + \ell\cos\theta). \tag{1}$$

Taking the derivatives of these coordinates, we see that the square of the speed is

$$V^2 = \dot{X}^2 + \dot{Y}^2 = \ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta. \tag{2}$$

The Lagrangian is therefore

$$L = K - U = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta) - mg(y + \ell\cos\theta). \tag{3}$$

The equation of motion for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \qquad \Longrightarrow \qquad \ell\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta. \tag{4}$$

Plugging in the explicit form of $y(t)$, we have

$$\ell\ddot{\theta} + \sin\theta\Big(A\omega^2\cos(\omega t) - g\Big) = 0. \tag{5}$$

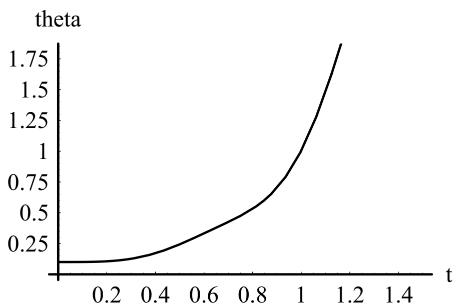
In retrospect, this makes sense. Someone in the reference frame of the support, which has acceleration $\ddot{y} = -A\omega^2\cos(\omega t)$, may as well be living in a world where the acceleration from gravity is $g - A\omega^2\cos(\omega t)$ downward. Eq. (5) is simply the $F = ma$ equation in the tangential direction in this accelerated frame.

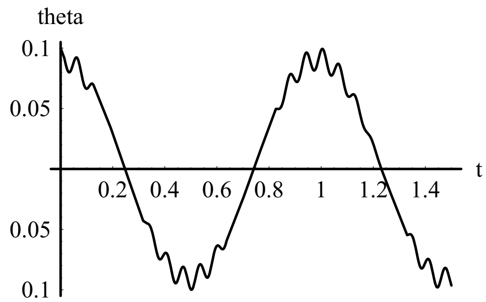
Assuming θ is small, we may set $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big) = 0, \tag{6}$$

where $\omega_0 \equiv \sqrt{g/\ell}$, and $a \equiv A/\ell$. Eq. (6) cannot be solved exactly, but we can still get a good idea of how θ depends on time. We can do this both numerically and (approximately) analytically.

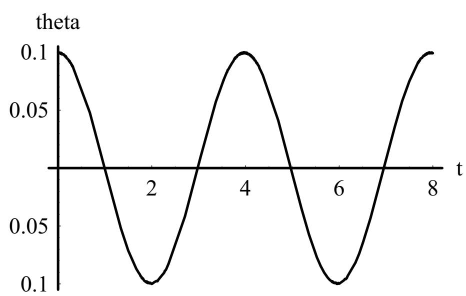
The figures below show how θ depends on time for parameters with values $\ell = 1\,\text{m}$, $A = 0.1\,\text{m}$, and $g = 10\,\text{m/s}^2$ (so $a = 0.1$, and $\omega_0^2 = 10\,\text{s}^{-2}$). In the first plot, $\omega = 10\,\text{s}^{-1}$. And in the second plot, $\omega = 100\,\text{s}^{-1}$. The stick falls over in first case, but undergoes oscillatory motion in the second case. Apparently, if ω is large enough the stick will not fall over.

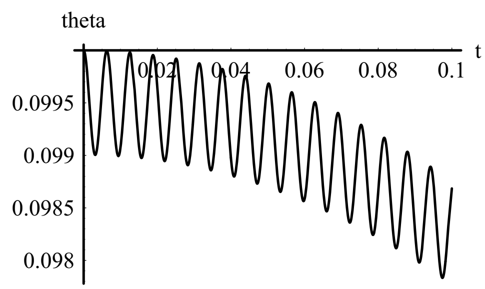




Let's now explain this phenomenon analytically. At first glance, it's rather surprising that the stick stays up. It seems like the average (over a few periods of the ω oscillations) of the tangential acceleration in eq. (6), namely $-\theta(a\omega^2\cos(\omega t)-\omega_0^2)$, equals the positive quantity $\theta\omega_0^2$, because the $\cos(\omega t)$ term averages to zero (or so it appears). So you might think that there is a net force making θ increase, causing the stick fall over.

The fallacy in this reasoning is that the average of the $-a\omega^2\theta\cos(\omega t)$ term is *not* zero, because θ undergoes tiny oscillations with frequency ω, as seen below. Both of these plots have $a = 0.005$, $\omega_0^2 = 10\,\mathrm{s}^{-2}$, and $\omega = 1000\,\mathrm{s}^{-1}$ (we'll work with small a and large ω from now on; more on this below). The second plot is a zoomed-in version of the first one near $t = 0$.





The important point here is that the tiny oscillations in θ shown in the second plot are correlated with $\cos(\omega t)$. It turns out that the θ value at the t where $\cos(\omega t) = 1$ is larger than the θ value at the t where $\cos(\omega t) = -1$. So there is a net negative contribution to the $-a\omega^2\theta\cos(\omega t)$ part of the acceleration. And it may indeed be large enough to keep the pendulum up, as we will now show.

To get a handle on the $-a\omega^2\theta\cos(\omega t)$ term, let's work in the approximation where ω is large and $a \equiv A/\ell$ is small. More precisely, we will assume $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, for reasons we will explain below. Look at one of the little oscillations in the second of the above plots. These oscillations have frequency ω, because they are due simply to the support moving up and down. When the support moves up, θ increases; and when the support moves down, θ decreases. Since the average position of the pendulum doesn't change much over one of these small periods, we can look for an approximate solution to eq. (6) of the form

$$\theta(t) \approx C + b\cos(\omega t), \tag{7}$$

where $b \ll C$. C will change over time, but on the scale of $1/\omega$ it is essentially constant, if $a \equiv A/\ell$ is small enough.

Plugging this guess for θ into eq. (6), and using $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, we find that $-b\omega^2\cos(\omega t) + Ca\omega^2\cos(\omega t) = 0$, to leading order.[1] So we must have $b = aC$. Our approximate solution for θ is therefore

$$\theta \approx C\Big(1 + a\cos(\omega t)\Big). \tag{8}$$

Let's now determine how C gradually changes with time. From eq. (6), the average acceleration of θ, over a period $T = 2\pi/\omega$, is

$$\begin{aligned}
\overline{\ddot{\theta}} &= \overline{-\theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&\approx \overline{-C\Big(1 + a\cos(\omega t)\Big)\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&= -C\Big(a^2\omega^2\overline{\cos^2(\omega t)} - \omega_0^2\Big) \\
&= -C\left(\frac{a^2\omega^2}{2} - \omega_0^2\right) \\
&\equiv -C\Omega^2,
\end{aligned} \tag{9}$$

where

$$\Omega = \sqrt{\frac{a^2\omega^2}{2} - \frac{g}{\ell}}\,. \tag{10}$$

But if we take two derivatives of eq. (7), we see that $\overline{\ddot{\theta}}$ simply equals $\ddot{C}$. Equating this value of $\overline{\ddot{\theta}}$ with the one in eq. (9) gives

$$\ddot{C}(t) + \Omega^2 C(t) \approx 0. \tag{11}$$

This equation describes nice simple-harmonic motion. Therefore, C oscillates sinusoidally with the frequency Ω given in eq. (10). This is the overall back and forth motion seen in the first of the above plots. Note that we must have $a\omega > \sqrt{2}\omega_0$ if this frequency is to be real so that the pendulum stays up. Since we have assumed $a \ll 1$, we see that $a^2\omega^2 > 2\omega_0^2$ implies $a\omega^2 \gg \omega_0^2$, which is consistent with our initial assumption above.

If $a\omega \gg \omega_0$, then eq. (10) gives $\Omega \approx a\omega/\sqrt{2}$. Such is the case if we change the setup and simply have the pendulum lie flat on a horizontal table where the acceleration from gravity is zero. In this limit where g is irrelevant, dimensional analysis implies that the frequency of the C oscillations must be a multiple of ω, because ω is the only quantity in the problem with units of frequency. It just so happens that the multiple is $a/\sqrt{2}$.

[1] The reasons for the $a \ll 1$ and $a\omega^2 \gg \omega_0^2$ qualifications are the following. If $a\omega^2 \gg \omega_0^2$, then the $a\omega^2\cos(\omega t)$ term dominates the ω_0^2 term in eq. (6). The one exception to this is when $\cos(\omega t) \approx 0$, but this occurs for a negligibly small amount of time if $a\omega^2 \gg \omega_0^2$. If $a \ll 1$, then we can legally ignore the $\ddot{C}$ term when eq. (7) is substituted into eq. (6). We will find below, in eq. (9), that our assumptions lead to $\ddot{C}$ being roughly proportional to $a^2\omega^2$. Since the other terms in eq. (6) are proportional to $a\omega^2$, we need $a \ll 1$ in order for the $\ddot{C}$ term to be negligible. In short, $a \ll 1$ is the condition under which C varies slowly on the time scale of $1/\omega$.