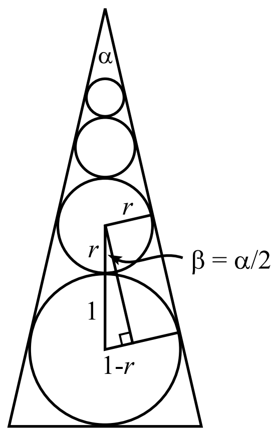
Solution

Week 68 (12/29/03)

Tower of circles

Let the bottom circle have radius 1, and let the second circle have radius r. From the following figure, we have

$$\sin\beta = \frac{1-r}{1+r}\,, \qquad \text{where } \beta \equiv \alpha/2. \tag{1}$$



In solving this problem, it is easier to work with r, instead of the angle α. So we will find the value of r for which A_C/A_T is maximum, and then use eq. (1) to obtain α. Note that r is the ratio of the radii of any two adjacent circles. This follows from the fact that we could have drawn the above thin little right triangle by using any two adjacent circles. Alternatively, it follows from the fact that if we scale up the top $N-1$ circles by the appropriate factor, then we obtain the bottom $N-1$ circles.

The area, A_T, of the triangle may be calculated in terms of r and N as follows. Since we could imagine stacking an infinite number of circles up to the vertex of the triangle, we see that the the height of the triangle is

$$h = 2 + 2r + 2r^2 + 2r^3 + \cdots = \frac{2}{1-r}\,. \tag{2}$$

The length of half the base, b, of the triangle is give by $b/2 = h\tan\beta$. But from eq. (1) we have $\tan\beta = (1-r)/(2\sqrt{r})$. Therefore,

$$b = \frac{2h}{\tan\beta} = \frac{2}{\sqrt{r}}\,. \tag{3}$$

The area of the triangle is then

$$A_T = \frac{bh}{2} = \frac{2}{(1-r)\sqrt{r}}\,. \tag{4}$$

The total area of the circles is

$$A_C = \pi\Big(1 + r^2 + r^4 + \cdots r^{2(N-1)}\Big) \tag{5}$$

$$= \pi\frac{1 - r^{2N}}{1 - r^2}\,. \tag{6}$$

Therefore, the ratio of the areas is

$$\frac{A_C}{A_T} = \frac{\pi}{2}\frac{\sqrt{r}(1 - r^{2N})}{1 + r}\,. \tag{7}$$

Setting the derivative of this equal to zero to obtain the maximum, we find

$$(1 - r) - (4N + 1)r^{2N} - (4N - 1)r^{2N+1} = 0. \tag{8}$$

In general, this can only be solved numerically for r. But if N is very large, we can obtain an approximate solution. To leading order in N, we may set $4N \pm 1 \approx 4N$. We may also set $r^{2N+1} \approx r^{2N}$, because r must be very close to 1 (otherwise there would be nothing to cancel the "1" term in eq. (8)). For convenience, let us write $r \equiv 1 - \epsilon$, where ϵ is very small. Eq. (8) then yields

$$8N(1 - \epsilon)^{2N} \approx \epsilon. \tag{9}$$

But $(1 - \epsilon)^{2N} \approx e^{-2N\epsilon}$.[1] Hence,

$$e^{-2N\epsilon} \approx \frac{\epsilon}{8N}\,. \tag{10}$$

Taking the log of both sides gives

$$\begin{aligned} \epsilon &\approx \frac{1}{2N}\ln\left(\frac{8N}{\epsilon}\right) \\ &\approx \frac{1}{2N}\ln\left(\frac{8N}{\frac{1}{2N}\ln\left(\frac{8N}{\epsilon}\right)}\right), \qquad \text{etc.} \end{aligned} \tag{11}$$

Therefore, to leading order in N, we have

$$\epsilon \approx \frac{1}{2N}\ln\left(\frac{16N^2}{\mathcal{O}(\ln N)}\right) = \frac{\ln N - \mathcal{O}(\ln\ln N) + \cdots}{N} \approx \frac{\ln N}{N}\,. \tag{12}$$

Note that for large N, this ϵ is much less than $1/\sqrt{N}$, so eq. (10) is indeed valid. Hence, $r = 1 - \epsilon \approx 1 - (\ln N)/N$. Eq. (1) then gives

$$\alpha = 2\beta \approx 2\sin\beta = 2\frac{1 - r}{1 + r} \approx \frac{2\epsilon}{2} = \epsilon, \tag{13}$$

and so

$$\alpha \approx \frac{\ln N}{N}\,. \tag{14}$$

This is the desired answer to leading order in N, in the sense that as N becomes very large, this answer becomes multiplicatively arbitrarily close to the true answer.

[1] This follows from taking the log of $(1-\epsilon)^{2N}$, to obtain $\ln((1-\epsilon)^{2N}) = 2N\ln(1-\epsilon) \approx -2N(\epsilon + \epsilon^2/2 + \cdots)$. This is approximately equal to $-2N\epsilon$, provided that the second term in the expansion is small, which is the case when $\epsilon \ll 1/\sqrt{N}$, which we will find to be true.

Remarks:

1. The radius of the top circle in the stack is

$$R_N = r^{N-1} \approx r^N = (1-\epsilon)^N \approx e^{-\ln N \epsilon}. \tag{15}$$

Using eq. (10) and then eq. (12), we have

$$R_N \approx \sqrt{\frac{\epsilon}{8N}} \approx \frac{\sqrt{\ln N}}{2\sqrt{2}N}\,. \tag{16}$$

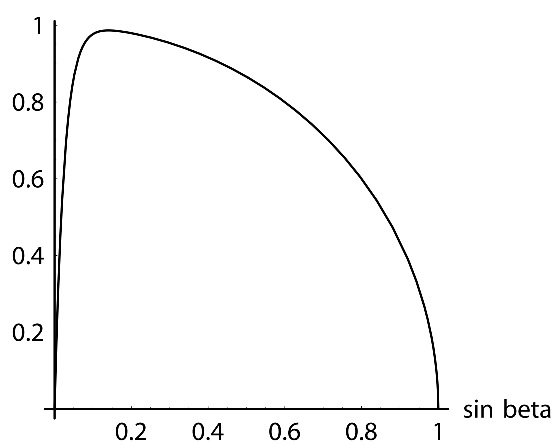
2. The distance from the center of the top circle to the vertex equals

$$\frac{R_N}{\sin\beta} \approx \frac{R_N}{\beta} \approx \frac{\frac{\sqrt{\ln N}}{2\sqrt{2}N}}{\frac{1}{2N}\ln N} = \frac{1}{\sqrt{2\ln N}}\,, \tag{17}$$

which goes to zero (but very slowly) for large N.

3. For $r \approx 1 - (\ln N)/N$, eq. (7) yields $A_C/A_T \approx \pi/4$. This is the expected answer, because if we look at a small number of adjacent circles, they appear to be circles inside a rectangle (because the long sides of the isosceles triangle are nearly parallel for small α), and it is easy to see that $\pi/4$ is the answer for the rectangular case.

4. Using eq. (7), along with $r = (1-\sin\beta)/(1+\sin\beta)$ from eq. (1), we can make a plot of $(4/\pi)(A_C/A_T)$ as a function of $\sin\beta$. The figure below shows the plot for $N = 20$. In the limit of very large N, the left part of the graph approaches a vertical segment. The rest of the curve approaches a quarter circle, as N goes to infinity. That is, $(4/\pi)(A_C/A_T) \approx \cos\beta$, for $N \to \infty$. This is true because if N is large, and if β is larger than order $(1/N)\ln N$, then we effectively have an infinite number of circles in the triangle. In this infinite case, the ratio A_C/A_T is given by the ratio of the area of a circle to the area of a circumscribing trapezoid whose sides are tilted at an angle β. You can show that this ratio is $(\pi/4)\cos\beta$.



5. We can also consider the more general case of higher dimensions. For example, instead of stacking N circles inside a triangle, we can stack N spheres inside a cone. Let α be the angle at the peak of the cone. Then the α for which the ratio of the total volume of the spheres to the volume of the cone is maximum is $\alpha \approx (2\ln N)/(3N)$. And the answer in the general case of d dimensions (with $d \geq 2$) is $\alpha \approx (2\ln N)/(dN)$. This agrees with eq. (14) for the $d = 2$ case. We can show this general result as follows.

As in the original problem, the height and base radius of the generalized "cone" are still

$$h = \frac{2}{1-r}, \qquad \text{and} \qquad b = \frac{2}{\sqrt{r}}. \tag{18}$$

Therefore, the "volume" of the cone is proportional to

$$V_{\text{cone}} \propto b^{d-1}h \propto \frac{1}{r^{(d-1)/2}(1-r)}. \tag{19}$$

The total volume of the "spheres" is proportional to

$$\begin{aligned} V_{\text{spheres}} &\propto 1 + r^d + r^{2d} + \cdots r^{(N-1)d} && (20) \\ &= \frac{1-r^{Nd}}{1-r^d}. && (21) \end{aligned}$$

Therefore,

$$\frac{V_{\text{spheres}}}{V_{\text{cone}}} \propto \frac{(1-r^{Nd})r^{(d-1)/2}(1-r)}{1-r^d}. \tag{22}$$

To maximize this, it is easier to work with the small quantity $\epsilon \equiv 1 - r$. In terms of ϵ, we have (using the binomial expansion)

$$\begin{aligned} \frac{V_{\text{spheres}}}{V_{\text{cone}}} &\propto \frac{(1-(1-\epsilon)^{Nd})(1-\epsilon)^{(d-1)/2}\epsilon}{1-(1-\epsilon)^d}. \\ &\approx \frac{(1-e^{-Nd\epsilon})\left[1-\left(\frac{d-1}{2}\right)\epsilon + \left(\frac{(d-1)(d-3)}{8}\right)\epsilon^2 - \cdots\right]}{d\left[1-\left(\frac{d-1}{2}\right)\epsilon + \left(\frac{(d-1)(d-2)}{6}\right)\epsilon^2 - \cdots\right]}. && (23) \end{aligned}$$

The terms in the square brackets in the numerator and the denominator differ at order ϵ^2, so we have

$$\frac{V_{\text{spheres}}}{V_{\text{cone}}} \propto (1-e^{-Nd\epsilon})(1 - A\epsilon^2 + \cdots), \tag{24}$$

where A happens to equal $(d^2 - 1)/24$, but we won't need this. Taking the derivative of eq. (24) with respect to ϵ to obtain the maximum, we find

$$(1-e^{-Nd\epsilon})(-2A\epsilon) + Nde^{-Nd\epsilon}(1-A\epsilon^2) = 0. \tag{25}$$

The N in the second term tells us that $e^{-Nd\epsilon}$ must be at most order $1/N$. Therefore, we can set $1 - e^{-Nd\epsilon} \approx 1$ in the first term. Also, we can set $1 - A\epsilon^2 \approx 1$ in the second term. This gives

$$\begin{aligned} e^{-Nd\epsilon} &\approx \frac{2A\epsilon}{Nd} \\ \Longrightarrow \qquad \epsilon &\approx \frac{1}{Nd}\ln\left(\frac{Nd}{2A\epsilon}\right). && (26) \end{aligned}$$

In the same manner as in eq. (12), we find, to leading order in N,

$$\epsilon \approx \frac{2\ln N}{dN}\,. \tag{27}$$

And since $\alpha = \epsilon$ from eq. (13), we obtain $\alpha \approx (2\ln N)/(dN)$, as desired.

6. Eq. (8) can be solved numerically for r, for any value of N. A few results are:

N	r	α (deg)	α (rad)	$(\ln N)/N$	$(4/\pi)(A_C/A_T)$
1	.333	60	1.05	0	.770
2	.459	43.6	.760	.347	.887
3	.539	34.9	.609	.366	.931
10	.754	16.1	.282	.230	.987
100	.953	2.78	.0485	.0461	.999645
1000	.9930	.400	$6.98 \cdot 10^{-3}$	$6.91 \cdot 10^{-3}$	$1 - 6.96 \cdot 10^{-6}$
10^6	.9999864	$7.76 \cdot 10^{-4}$	$1.36 \cdot 10^{-5}$	$1.38 \cdot 10^{-5}$	$1 - 2.47 \cdot 10^{-11}$

For large N, we see that

$$\alpha \text{ (rad)} \approx \frac{\ln N}{N}\,, \qquad \text{and} \qquad r \approx 1 - \frac{\ln N}{N}\,. \tag{28}$$

Also, using eqs. (7) and (12), you can show that to leading order in N,

$$\frac{4}{\pi}\frac{A_C}{A_T} \approx 1 - \frac{(\ln N)^2 + \ln N}{8N^2}\,, \tag{29}$$

which agrees well with the numerical results, for large N.