Solution

Week 69 (1/5/04)

Compton scattering

We will solve this problem by making use of 4-momenta. The *4-momentum* of a particle is given by

$$P \equiv (P_0, P_1, P_2, P_3) \equiv (E, p_x c, p_y c, p_z c) \equiv (E, \mathbf{p}c). \tag{1}$$

In general, the inner-product of two *4-vectors* is given by

$$A \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3. \tag{2}$$

The square of a 4-momentum (that is, the inner product of a 4-momentum with itself) is therefore

$$P^2 \equiv P \cdot P = E^2 - |\mathbf{p}|^2 c^2 = m^2 c^4. \tag{3}$$

Let's now apply these idea to the problem at hand. We will actually be doing nothing here other than applying conservation of energy and momentum. It's just that the language of 4-vectors makes the whole procedure surprisingly simple. Note that conservation of E and $\mathbf{p}$ during the collision can be succinctly written as

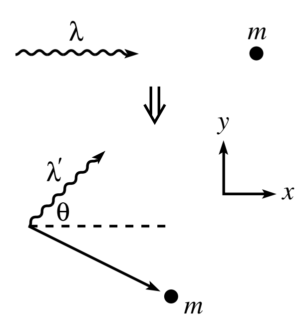
$$P_{\text{before}} = P_{\text{after}}. \tag{4}$$

Referring to the figure below, the 4-momenta before the collision are

$$P_\gamma = \left(\frac{hc}{\lambda}, \frac{hc}{\lambda}, 0, 0\right), \qquad P_m = (mc^2, 0, 0, 0). \tag{5}$$

And the 4-momenta after the collision are

$$P'_\gamma = \left(\frac{hc}{\lambda'}, \frac{hc}{\lambda'}\cos\theta, \frac{hc}{\lambda'}\sin\theta, 0\right), \qquad P'_m = (\text{we won't need this}). \tag{6}$$



If we wanted to, we could write P'_m in terms of its momentum and scattering angle. But the nice thing about this 4-momentum method is that we don't need to introduce any quantities that we're not interested in.

Conservation of energy and momentum give $P_\gamma + P_m = P'_\gamma + P'_m$. Therefore,

$$\begin{aligned}
(P_\gamma + P_m - P'_\gamma)^2 &= P'^2_m \\
\Longrightarrow P^2_\gamma + P^2_m + P'^2_\gamma + 2P_m(P_\gamma - P'_\gamma) - 2P_\gamma P'_\gamma &= P'^2_m \\
\Longrightarrow 0 + m^2c^4 + 0 + 2mc^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}(1-\cos\theta) &= m^2c^4. \qquad (7)
\end{aligned}$$

Multiplying through by $\lambda\lambda'/(2hmc^3)$ gives the desired result,

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta). \qquad (8)$$

The ease of this solution arose from the fact that all the unknown garbage in P'_m disappeared when we squared it.

Remarks:

1. If $\theta \approx 0$ (that is, not much scattering), then $\lambda' \approx \lambda$, as expected.
2. If $\theta = \pi$ (that is, backward scattering) and additionally $\lambda \ll h/mc$ (that is, $mc^2 \ll hc/\lambda = E_\gamma$), then $\lambda' \approx 2h/mc$, so

$$E'_\gamma = \frac{hc}{\lambda'} \approx \frac{hc}{\frac{2h}{mc}} = \frac{1}{2}mc^2. \qquad (9)$$

Therefore, the photon bounces back with an essentially fixed E'_γ, independent of the initial E_γ (as long as E_γ is large enough). This isn't all that obvious.