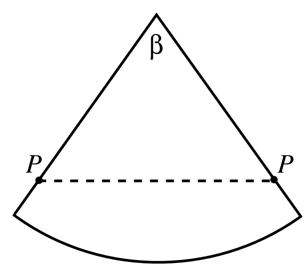
Solution

Week 7 (10/28/02)

Mountain climber

(a) We will take advantage of the fact that a cone is "flat", in the sense that we can make one out of a piece of paper, without crumpling the paper.

Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, which is a sector of a circle, S.



If the cone is very sharp, then S will look like a thin "pie piece". If the cone is very wide, with a shallow slope, then S will look like a pie with a piece taken out of it. Points on the straight-line boundaries of the sector S are identified with each other. Let P be the location of the lasso's knot. Then P appears on each straight-line boundary, at equal distances from the tip of S. Let β be the angle of the sector S.

The key to this problem is to realize that the path of the lasso's loop must be a straight line on S, as shown in the above figure. (The rope will take the shortest distance between two points since there is no friction. And rolling the cone onto a plane does not change distances.) A straight line between the two identified points P is possible if and only if the sector S is smaller than a semicircle. The restriction on the mountain is therefore $\beta < 180°$.

What is this restriction, in terms of the angle of the peak, α? Let C denote a cross-sectional circle, a distance d (measured along the cone) from the top of the mountain. A semicircular S implies that the circumference of C equals πd. This then implies that the radius of C equals $d/2$. Therefore,

$$\sin(\alpha/2) < \frac{d/2}{d} = \frac{1}{2} \qquad \Longrightarrow \qquad \alpha < 60° \tag{1}$$

is the condition under which the mountain is climbable. In short, having $\alpha < 60°$ guarantees that there is a loop around the cone of shorter length than the distance straight to the peak and back.

Remark: When viewed from the side, the rope will appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies $\alpha < 90°$. This is not the case, because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop; but the loop must certainly have a kink in it where the knot is, since there must exist a vertical component to the tension. (If we had posed the problem with a planar, triangular mountain, then the answer would be $\alpha < 90°$.)

(b) Use the same strategy. Roll the cone onto a plane. If the mountain is very steep, then the climber's position can fall by means of the loop growing larger. If the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the sector S in a plane, this condition requires that if we move P a distance ℓ up (or down) along the mountain, the distance between the identified points P decreases (or increases) by ℓ. In the above figure, we must therefore have an equilateral triangle, and so $\beta = 60°$.

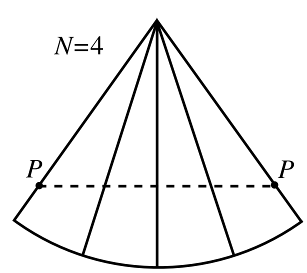
What peak-angle α does this correspond to? As above, let C be a cross-sectional circle, a distance d (measured along the cone) from the top of the mountain. Then $\beta = 60°$ implies that the circumference of C equals $(\pi/3)d$. This then implies that the radius of C equals $d/6$. Therefore,

$$\sin(\alpha/2) = \frac{d/6}{d} = \frac{1}{6} \qquad \Longrightarrow \qquad \alpha \approx 19° \tag{2}$$

is the condition under which the mountain is climbable. We see that there is exactly one angle for which the climber can climb up along the mountain.

REMARK: Another way to see the $\beta = 60°$ result is to note that the three directions of rope emanating from the knot must all have the same tension, since the deluxe lasso is one continuous piece of rope. They must therefore have $120°$ angles between themselves (to provide zero net force on the massless knot). This implies that $\beta = 60°$.

FURTHER REMARKS: For each type of lasso, for what angles can the mountain be climbed if the lasso is looped N times around the top of the mountain? The solution here is similar to that above. For the "cheap" lasso of part (a), roll the cone N times onto a plane, as shown for $N = 4$.



The resulting figure S_N is a sector of a circle divided into N equal sectors, each representing a copy of the cone. As above, S_N must be smaller than a semicircle. The circumference of the circle C (defined above) must therefore be less than $\pi d/N$. Hence, the radius of C must be less than $d/2N$. Thus,

$$\sin(\alpha/2) < \frac{d/2N}{d} = \frac{1}{2N} \qquad \Longrightarrow \qquad \alpha < 2\sin^{-1}\left(\frac{1}{2N}\right). \tag{3}$$

For the “deluxe” lasso of part (b), again roll the cone N times onto a plane. From the reasoning in part (b), we must have $N\beta = 60°$. The circumference of C must therefore be $\pi d/3N$, and so its radius must be $d/6N$. Therefore,

$$\sin(\alpha/2) = \frac{d/6N}{d} = \frac{1}{6N} \qquad \Longrightarrow \qquad \alpha = 2\sin^{-1}\left(\frac{1}{6N}\right). \tag{4}$$