Solution

Week 70 (1/12/04)

Painting a funnel

It is true that the volume is finite, and that you can fill it up with paint. It is also true that the surface area is infinite, but you actually *can* paint it.

The apparent paradox arises from essentially comparing apples and oranges. In our case we are comparing areas (things of dimension two) with volumes (things of dimension three). When someone says that the funnel can't be painted, she is saying that it would take an infinite *volume* of paint to cover it. But the fact that the surface area is infinite does *not* imply that it takes an infinite volume of paint to cover it. To be sure, if we try to paint the funnel with a given fixed thickness of paint, then we would indeed need an infinite volume of paint. But is this case we would essentially have a tube of paint of fixed radius, for large values of x, with the funnel taking up a negligible volume at the center of the tube. This tube certainly has an infinite volume.

But what if we paint the funnel with a decreasing thickness of paint, as x gets larger? For example, if we make the thickness go like $1/x$, then the volume of paint goes like $\int_1^\infty (1/x^2)\,dx$, which is finite. In this manner, we can indeed paint the funnel. To sum up, we buy paint by the gallon, not by the square meter. And a gallon of paint can cover an infinite area, as long as we make the thickness go to zero fast enough.