Solution

Week 71 (1/19/04)

Maximum trajectory length

Let θ be the angle at which the ball is thrown. Then the coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. The ball reaches its maximum height at $t = v\sin\theta/g$, so the length of the trajectory is

$$\begin{aligned} L &= 2\int_0^{v\sin\theta/g} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \\ &= 2\int_0^{v\sin\theta/g} \sqrt{(v\cos\theta)^2 + (v\sin\theta - gt)^2}\, dt \\ &= 2v\cos\theta \int_0^{v\sin\theta/g} \sqrt{1 + \left(\tan\theta - \frac{gt}{v\cos\theta}\right)^2}\, dt. \qquad (1) \end{aligned}$$

Letting $z \equiv \tan\theta - gt/v\cos\theta$, we obtain

$$L = -\frac{2v^2\cos^2\theta}{g}\int_{\tan\theta}^0 \sqrt{1+z^2}\, dz. \qquad (2)$$

Letting $z \equiv \tan\alpha$, and switching the order of integration, gives

$$L = \frac{2v^2\cos^2\theta}{g}\int_0^\theta \frac{d\alpha}{\cos^3\alpha}. \qquad (3)$$

You can either look up this integral, or you can derive it (see the remark at the end of the solution). The result is

$$\begin{aligned} L &= \frac{2v^2\cos^2\theta}{g}\cdot\frac{1}{2}\left(\frac{\sin\theta}{\cos^2\theta} + \ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\right) \\ &= \frac{v^2}{g}\left(\sin\theta + \cos^2\theta\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)\right). \qquad (4) \end{aligned}$$

As a double-check, you can verify that $L = 0$ when $\theta = 0$, and $L = v^2/g$ when $\theta = 90^\circ$. Taking the derivative of eq. (4) to find the maximum, we obtain

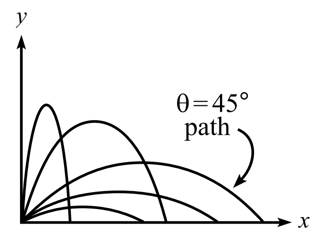
$$0 = \cos\theta - 2\cos\theta\sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + \cos^2\theta\left(\frac{\cos\theta}{1+\sin\theta}\right)\frac{\cos^2\theta + (1+\sin\theta)\sin\theta}{\cos^2\theta}. \qquad (5)$$

This reduces to

$$1 = \sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right). \qquad (6)$$

Finally, you can show numerically that the solution for θ is $\theta_0 \approx 56.5^\circ$.

A few possible trajectories are shown below. Since it is well known that $\theta = 45^\circ$ provides the maximum *horizontal* distance, it follows from the figure that the θ_0 yielding the arc of maximum *length* must satisfy $\theta_0 \geq 45^\circ$. The exact angle, however, requires the above detailed calculation.



Remark: Let's now show that the integral in eq. (3) is given by

$$\int \frac{d\alpha}{\cos^3\alpha} = \frac{1}{2}\left(\frac{\sin\alpha}{\cos^2\alpha} + \ln\left(\frac{1+\sin\alpha}{\cos\alpha}\right)\right). \tag{7}$$

Letting $c \equiv \cos\alpha$ and $s \equiv \sin\alpha$ for convenience, and dropping the $d\alpha$ in the integrals, we have

$$\begin{aligned}
\int \frac{1}{c^3} &= \int \frac{c}{c^4} \\
&= \int \frac{c}{(1-s^2)^2} \\
&= \frac{1}{4}\int c\left(\frac{1}{1+s} + \frac{1}{1-s}\right)^2 \\
&= \frac{1}{4}\int \left(\frac{c}{(1+s)^2} + \frac{c}{(1-s)^2}\right) + \frac{1}{2}\int \frac{c}{(1-s^2)} \\
&= \frac{1}{4}\left(\frac{-1}{1+s} + \frac{1}{1-s}\right) + \frac{1}{4}\int \left(\frac{c}{1+s} + \frac{c}{1-s}\right) \\
&= \frac{s}{2(1-s^2)} + \frac{1}{4}\Big(\ln(1+s) - \ln(1-s)\Big) \\
&= \frac{s}{2c^2} + \frac{1}{4}\ln\left(\frac{1+s}{1-s}\right) \\
&= \frac{1}{2}\left(\frac{s}{c^2} + \ln\left(\frac{1+s}{c}\right)\right),
\end{aligned} \tag{8}$$

as we wanted to show.