Solution

Week 76 (2/23/04)

Crawling ant

At time t, the movable end of the band is a distance $\ell(t) = L + Vt$ from the wall. Let the ant's distance from the wall be $x(t)$, and consider the fraction of the length along the band, $F(t) = x(t)/\ell(t)$. The given question is equivalent to: For what value of t does the fraction $F(t)$ become zero (if at all)? To answer this, let us see how $F(t)$ changes with time.

After an infinitesimal time, dt, the ant's position, x, increases by $(x/\ell)V\,dt$ due to the stretching, and decreases by $u\,dt$ due to the crawling. Therefore,

$$\begin{aligned} F(t+dt) &= \frac{x + (x/\ell)V\,dt - u\,dt}{\ell + V\,dt} \\ &= \frac{x}{\ell} - \frac{u\,dt}{\ell + V\,dt}\,. \end{aligned} \tag{1}$$

To first order in dt, this yields

$$F(t+dt) = F(t) - \frac{u}{\ell}\,dt. \tag{2}$$

In other words, $F(t)$ decreases due to the fact that in a time dt the ant crawls a distance $u\,dt$ relative to the band, which has a length $\ell(t)$. Eq. (2) gives

$$\frac{dF(t)}{dt} = -\frac{u}{\ell}\,. \tag{3}$$

Using $\ell(t) = L + Vt$ and integrating eq. (3), we obtain

$$F(t) = 1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right), \tag{4}$$

where the constant of integration has been chosen to satisfy $F(0) = 1$. We now note that for *any* positive values of u and V, we can make $F(t) = 0$ by choosing

$$t = \frac{L}{V}\left(e^{V/u} - 1\right). \tag{5}$$

For large V/u, the time it takes the ant to reach the wall becomes exponentially large, but it does indeed reach it in a finite time. For small V/u, you can use $e^{\epsilon} \approx 1 + \epsilon$ to show that eq. (5) reduces to $t \approx L/u$, as it should.

Remark: If $u < V$, then the ant will initially get carried away from the wall before it eventually comes back and reaches it. What is the maximum distance it gets from the wall? The ant's distance from the wall is

$$x(t) = F(t)\ell(t) = \left(1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right)\right)\left(L + Vt\right). \tag{6}$$

Setting the derivative of this equal to zero gives

$$\left(1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right)\right)V - u = 0. \tag{7}$$

Note that we could have arrived at this by simply demanding that the speed of the ant be zero, which means that the $F(t)V$ speed due to the stretching cancels the u speed in the other direction due to the crawling. The fraction $F(t)$ is therefore simply u/V.

Solving eq. (7) for t gives

$$t_{\max} = \frac{L}{V}\left(e^{V/u-1} - 1\right). \tag{8}$$

This holds only if $V \geq u$. If $V < u$, then $t_{\max} = 0$ and $x_{\max} = L$. Plugging the $t_{\max}$ from eq. (8) into eq. (6) gives

$$x_{\max} = \frac{u}{V}\frac{L}{e}e^{V/u} \qquad (V \geq u). \tag{9}$$

If, for example, $V = 2u$, then $x_{\max} \approx (1.36)L$. And if $V = 10u$, then $x_{\max} \approx (800)L$.

Note that for large V/u, the $t_{\max}$ in eq. (8) is approximately $1/e$ times the time it takes the ant to reach the wall, given in eq. (5).