Solution

Week 77 (3/1/04)

Relativistic momentum paradox

The reasoning is not correct. The horizontal speed of the masses remains the same. The system does *not* slow down in the x-direction. We can see that this must be the case, by looking at the setup in the inertial frame that moves in the x-direction with the v_x that the masses have when they are still on the constraints (which happens to be the same v_x that they always have). In this moving frame, the masses feel a force only in the y-direction when they come off the constraints, so they won't move horizontally with respect to this frame. Therefore, they will always move with constant v_x in the lab frame.

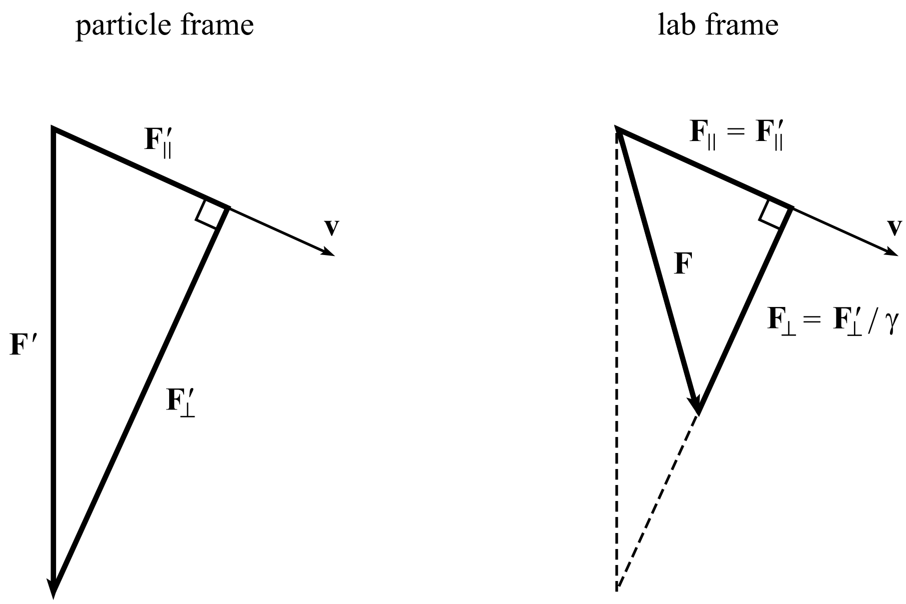
Since v_x is constant, the final p_x of the resulting blob is larger than the sum of the initial p_x's of the two masses, because the mass of the resulting blob is indeed greater than the sum of the initial masses. Where does the extra momentum come from? It comes from the string, which initially had some p_x because it had stored energy (because work was done to extend the string) and hence mass.

But where is the error in the reasoning stated in the problem? The first sentence is the one that is incorrect. In the lab frame, the force on each mass actually does *not* point in the y-direction. The reason for this is the following.

It turns out that in relativity, a transverse force is decreased by a factor γ when going from the particle's frame to another frame. (We'll just accept this fact here.) More precisely, consider an object flying past you in some direction. In the particle's frame (S'), let the force components be $F'_\parallel$ in that direction, and $F'_\perp$ in the orthogonal direction. Then the force components that you measure on the particle in the lab frame (S) are

$$F_\parallel = F'_\parallel \qquad \text{and} \qquad F_\perp = \frac{F'_\perp}{\gamma}\,. \tag{1}$$

Let's see what this implies. After the masses have been drawn together by the string a bit, they will be heading diagonally upward or downward, instead of directly in the x-direction. Let the upper mass be traveling in the direction $\mathbf{v}$ shown below. In the frame of the mass, the force $\mathbf{F}'$ points directly downward. But if we break this force into the components along the motion and perpendicular to the motion, we see that when transforming to the lab frame, the transverse component is decreased by a factor γ, as shown. The force $\mathbf{F}$ in the lab frame therefore points slightly forward, as shown. This forward component of the force is what increases the x-momentum of the masses. The x-momentum of each mass does indeed increase, because p_x takes the form $p_x = \gamma m v_x$, and γ increases due to the fact that the speed increases because of the increasing v_y.


particle frame
lab frame
$\mathbf{F}'_{||}$
$\mathbf{F}'$
$\mathbf{F}'_{\perp}$
$\mathbf{v}$
$\mathbf{F}_{||} = \mathbf{F}'_{||}$
$\mathbf{F}$
$\mathbf{F}_{\perp} = \mathbf{F}'_{\perp} / \gamma$
$\mathbf{v}$