Solution

Week 78 (3/8/04)

Infinite square roots

Let

$$x = \sqrt{1 - \sqrt{\frac{17}{16} - \sqrt{1 - \sqrt{\frac{17}{16} - \sqrt{1 - \cdots}}}}}. \qquad (1)$$

Then $x = \sqrt{1 - \sqrt{17/16 - x}}$. Squaring a few times yields $x^4 - 2x^2 + x - 1/16 = 0$, or equivalently,

$$(2x-1)(8x^3 + 4x^2 - 14x + 1) = 0. \qquad (2)$$

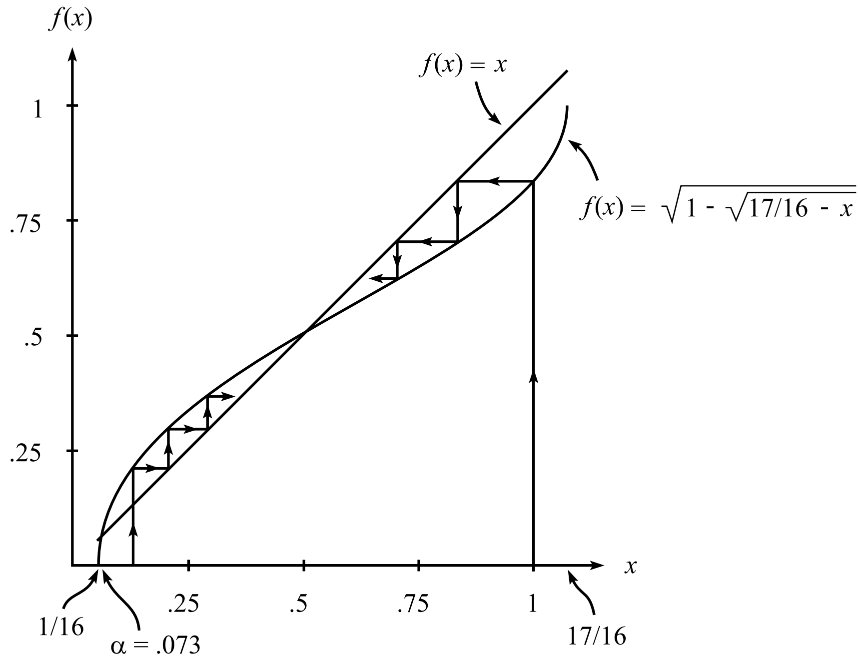
Therefore, either $x = 1/2$, or x is a root of $8x^3 + 4x^2 - 14x + 1 = 0$. Solving this cubic equation numerically, or fiddling around with the values at a few points, shows that it has three real roots. One is negative, between -1 and -2 (≈ -1.62). One is slightly greater than 1 (≈ 1.05). And one is slightly greater than $1/14$ $(\approx .073)$.

The first two of these cannot be the answer to the problem, because x must be positive and less than one. So the only possibilities are $x = 1/2$ and $x \approx .073$. A double-check shows that neither was introduced in the squaring steps above; both are correct solutions to $x = \sqrt{1 - \sqrt{17/16 - x}}$. But x clearly has one definite value, so which is it? A few iterations on a calculator, starting with a "1" under the innermost radical, shows that $x = 1/2$ is definitely the answer. Why?

The answer lies in the behavior of $f(x) = \sqrt{1 - \sqrt{17/16 - x}}$ near the points $x = 1/2$ and $x = \alpha \equiv .07318\ldots$. The point $x = 1/2$ is stable, in the sense that if we start with an x value slightly different from $1/2$, then $f(x)$ will be closer to $1/2$ than x was. The point $x = \alpha$, on the other hand, is unstable, in the sense that if we start with an x value slightly different from α, then $f(x)$ will be farther from α than x was.

Said in another way, the slope of $f(x)$ at $x = 1/2$ is less than 1 in absolute value (it is in fact $2/3$), while the slope at $x = \alpha$ is greater than 1 in absolute value (it is approximately 3.4). What this means is that points tend to head toward $1/2$, but away from α, under iteration by $f(x)$. In particular, if we start with the value 1, as we are supposed to do in this problem, then we will eventually get arbitrarily close to $1/2$ after many applications of f. Therefore, $1/2$ is the correct answer.

The above reasoning is perhaps most easily understood via the figure below. This figure shows graphically what happens to two initial values of x, under iteration by f. To find what happens to a given point x_0, draw a vertical line from x_0 to the curve $y = f(x)$; this gives $f(x_0)$. Then draw a horizontal line to the point $(f(x_0), f(x_0))$ on the line $y = x$. Then draw a vertical line to the curve $y = f(x)$; this gives $f(f(x_0))$. Continue drawing these horizontal and vertical lines to obtain successive iterations by f.



Using this graphical method, it is easy to see that if:

- $x_0 > 17/16$, then we immediately get imaginary values.
- $\alpha < x_0 \leq 17/16$, then iteration by f will lead to $1/2$.
- $x_0 = \alpha$ exactly, then we stay at α under iteration by f.
- $x_0 < \alpha$, then we will eventually get imaginary values. If $x_0 < 1/16$, then imaginary values will occur immediately, after one iteration; otherwise it will take more than one iteration.