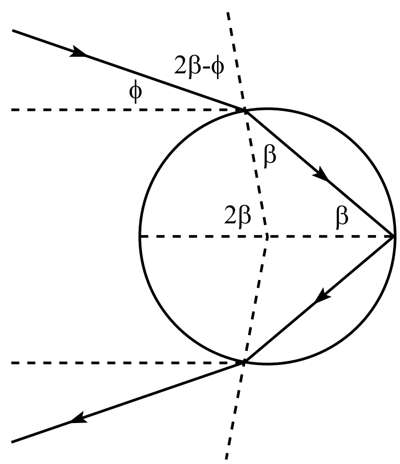
Solution

Week 81 (3/29/04)

Rainbows

Rainbows exist due to the fact that raindrops scatter light preferentially in certain directions. The effect of this "focusing" is to make the sky appear brighter in a certain region, as we will see in detail below. This brightness is the rainbow that you see. The colors of the rainbow are caused by the different indexes of refraction of the different colors; more on this below.

Primary rainbow: The preferred direction of the scattered light depends on how many internal reflections the light ray undergoes in the raindrop. Let's first consider the case of one internal reflection, as shown below. The light ray refracts into the raindrop, then reflects inside, and then refracts back out into the air. Let β and ϕ be defined as in the figure. Then the angles 2β and $2\beta - \phi$ are shown.



What is ϕ as a function of β? Snell's law at the refraction points gives

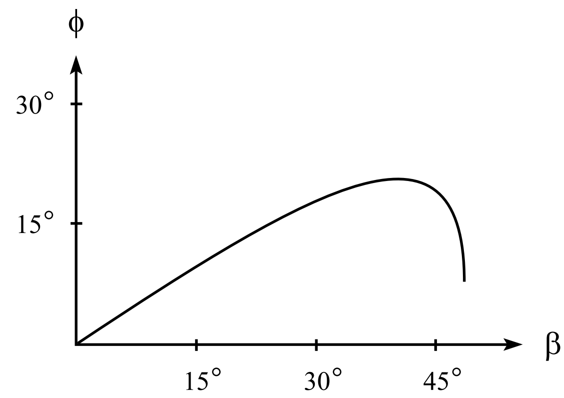
$$\sin(2\beta - \phi) = \frac{4}{3}\sin\beta. \tag{1}$$

Solving for ϕ yields

$$\phi = 2\beta - \arcsin\left(\frac{4}{3}\sin\beta\right). \tag{2}$$

The plot of ϕ vs. β looks like the following.[1]

[1]Note that β cannot be greater than $\arcsin(3/4) \approx 48.6°$, because this would yield the acrsin of a number larger than 1. $48.6°$ is the *critical angle* for the air/water interface.



We see that ϕ has a maximum of roughly $\phi_{\text{max}} \approx 20°$ at $\beta_{\text{max}} \approx 40°$. To be more precise, we can set $d\phi/d\beta = 0$. The result is

$$0 = 2 - \frac{\frac{4}{3}\cos\beta}{\sqrt{1 - \frac{16}{9}\sin^2\beta}}. \tag{3}$$

Squaring and using $\cos^2\beta = 1 - \sin^2\beta$ yields

$$\sin\beta_{\text{max}} = \sqrt{\frac{5}{12}} \qquad \Longrightarrow \qquad \beta_{\text{max}} \approx 40.2°. \tag{4}$$

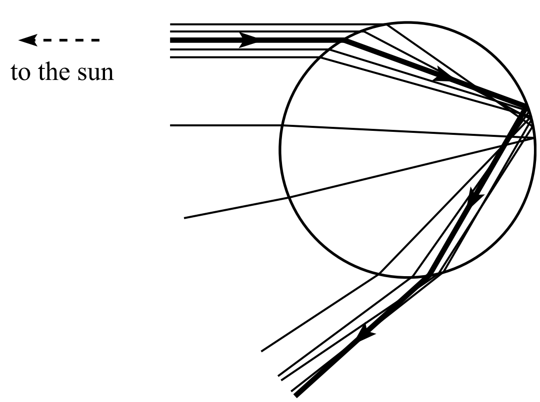
Substituting this back into eq. (2) gives

$$\phi_{\text{max}} \approx 21.0°. \tag{5}$$

The significance of this maximum is not that it is the largest value of ϕ, but rather that the slope of the $\phi(\beta)$ curve is zero, which means that there are many different values of β that yield essentially the same value ($\approx 21°$) of ϕ. The light therefore[2] gets "focused" into the total angle of

$$2\phi_{\text{max}} \approx 42°, \tag{6}$$

so the sky appears brighter at this angle (relative to the line from the sun to you). This brightness is the rainbow that you see. This reasoning is perhaps more clear if we draw a diagram with the incoming angle of the rays constant (which is in fact the case, because all of the sun's rays are parallel). A rough diagram of this is shown below. The bold line is the path with the maximum 2ϕ angle of 42°. You can see that the outgoing rays pile up at this angle.



[2]See the fourth remark below for elaboration on this reasoning.

Remarks:

1. The above reasoning explains why the rainbow is where it is. But why do we see the different colors? The colors arise from the fact that different wavelengths of light have different indexes of refraction at the air/water boundary. At one end of the visible spectrum, violet light has an index of 1.344. And at the other end, red light has an index of 1.332.[3] If you go through the above calculation, but now with these indexes in place of the "4/3" we used above, you will find that violet light appears at an angle of $2\phi_{\text{max}}^{\text{violet}} \approx 40.5^\circ$, and red light appears at an angle of $2\phi_{\text{max}}^{\text{red}} \approx 42.2^\circ$. Since red occurs at the larger angle, it is therefore the color at the top of the rainbow. Violet is at the bottom, and intermediate wavelengths are in between. The fact that red is at the top can be traced to the fact that 21° is the *maximum* value of ϕ.

2. A rainbow is actually a little wider than the approximately 2° spread we just found, because the sun isn't a point source. It subtends an angle of about half a degree, which adds half a degree to the rainbow's spread. Also, the rainbow's colors are somewhat washed out on the scale of half a degree, so a rainbow isn't as crisp as it would be if the sun were a point source.

3. In addition to the ordering of the colors, there is another consequence of the fact that 21° is the *maximum* value of ϕ. It is possible for a raindrop to scatter light at 2ϕ values smaller than 42° (for one internal reflection), but impossible for larger angles. Therefore, the region in the sky below the rainbow appears brighter than the region above it. Even though the focusing effect occurs only right at the rainbow, the simple scattering of light through 2ϕ values smaller than 42° makes the sky appear brighter below the rainbow.

4. There is one slight subtlety in the above reasoning we should address, even though turns out not to be important. We said above that if many different β values correspond to a certain value of ϕ, then there will be more light scattered into that value of ϕ. However, the important thing is not how many β values correspond to a certain ϕ, but rather the "cross section" of light that corresponds to that ϕ. Since the light hits the raindrop at an angle $2\beta - \phi$ with respect to the normal, the amount of light that corresponds to a given $d\beta$ interval is decreased by a factor of $\cos(2\beta - \phi)$. What we did in the above solution was basically note that the interval $d\beta$ that corresponds to a given interval $d\phi$ is

$$d\beta = \frac{d\phi}{d\phi/d\beta}\,, \tag{7}$$

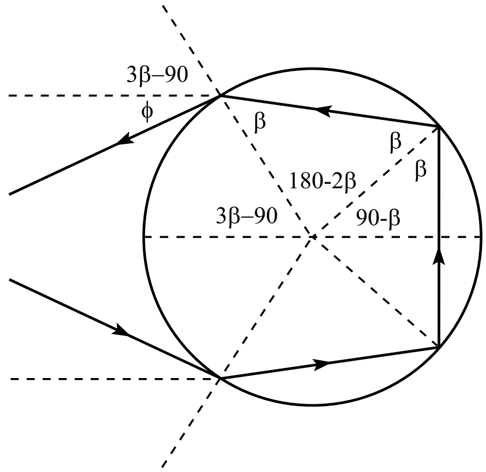
which diverges at ϕ_{max}, because $d\phi/d\beta = 0$ there. But since we are in fact concerned with the number of light rays that correspond to the interval $d\phi$, we now note that this number is proportional to

$$d\beta\cos(2\beta - \phi) = \frac{d\phi}{d\phi/d\beta}\cos(2\beta - \phi). \tag{8}$$

But $2\beta - \phi \neq 90^\circ$ at ϕ_{max}, so this still diverges at ϕ_{max}. Our answer therefore remains the same.

Secondary rainbow: Now consider the secondary rainbow. This rainbow arises from the fact that the light may undergo two reflections inside the raindrop before it refracts back out. This scenario is shown below. With β defined as in the figure, the $180^\circ - 2\beta$ and $90^\circ - \beta$ angles follow, and then the $3\beta - 90^\circ$ angle follows.

[3] The ends of the visible spectrum are somewhat nebulous, but I think these values are roughly correct.



Snell's law at the refraction points gives

$$\sin(3\beta - 90^\circ + \phi) = \frac{4}{3}\sin\beta. \tag{9}$$

Solving for ϕ yields

$$\phi = 90^\circ - 3\beta + \arcsin\left(\frac{4}{3}\sin\beta\right). \tag{10}$$

Taking the derivative to find the extremum (which is a minimum this time) gives

$$\sin\beta_{\text{min}} = \sqrt{\frac{65}{128}} \qquad \Longrightarrow \qquad \beta_{\text{min}} \approx 45.4^\circ. \tag{11}$$

Substituting this back into eq. (10) gives

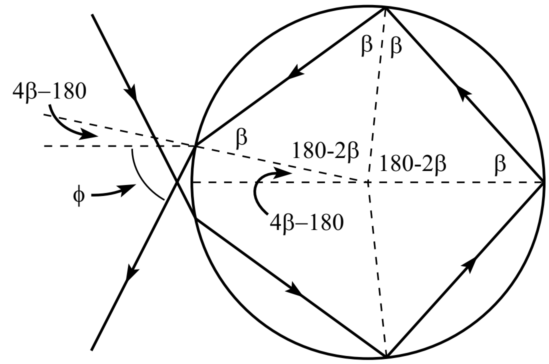
$$\phi_{\text{min}} \approx 25.4^\circ. \tag{12}$$

Using the same reasoning as above, we see that the light gets focused into the total angle of $2\phi_{\text{max}} \approx 51^\circ$, so the the sky appears brighter at this angle.

The fact that $\phi(\beta)$ has a minimum instead of a maximum means that the rainbow is inverted, with violet now on top. More precisely, if you go through the above calculation, but now with the red and violet indexes of refraction in place of the "4/3" used above, you will find that violet light appears at an angle of 53.8°, and red light appears at an angle of 50.6°. The spread here is a little larger than that for the primary rainbow above.

The secondary rainbow is fainter than the primary one for three reasons. First, the larger spread means that the light is distributed over a larger area. Second, additional light is lost at the second internal reflection. And third, the angle with respect to the normal at which the light ray hits the raindrop is larger for the secondary rainbow. For the primary rainbow it was $2\beta - \phi \approx 59^\circ$, while for the secondary rainbow it is $3\beta - 90^\circ + \phi \approx 72^\circ$.

Tertiary rainbow: Now consider the tertiary rainbow. This rainbow arises from three reflections inside the raindrop, as shown below. With β defined as in the figure, the two $180^\circ - 2\beta$ angles follow, and then the $4\beta - 180^\circ$ angle follows.



Snell's law at the refraction points gives

$$\sin(4\beta - 180^\circ + \phi) = \frac{4}{3}\sin\beta. \tag{13}$$

Solving for ϕ yields

$$\phi = 180^\circ - 4\beta + \arcsin\left(\frac{4}{3}\sin\beta\right). \tag{14}$$

Taking the derivative to find the extremum (which is a minimum) gives

$$\sin\beta_{\text{min}} = \sqrt{\frac{8}{15}} \qquad \Longrightarrow \qquad \beta_{\text{min}} \approx 46.9^\circ. \tag{15}$$

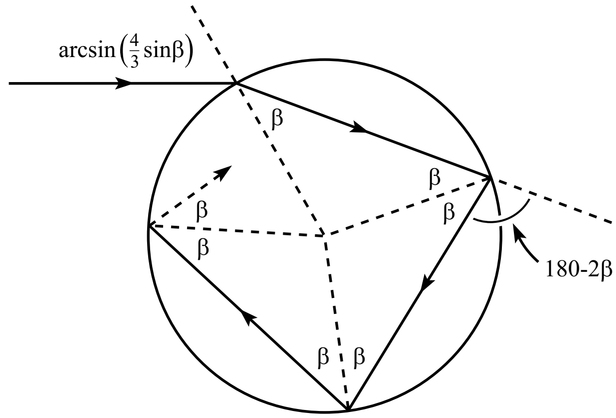
Substituting this back into eq. (10) gives

$$\phi_{\text{min}} \approx 69.2^\circ. \tag{16}$$

Using the same reasoning as above, we see that the light gets focused into the total angle of $2\phi_{\text{max}} \approx 138^\circ$, so the the sky appears brighter at this angle. The fact that this angle is larger than 90° means that the rainbow is actually behind you (if you are facing the primary and secondary rainbows), back towards the sun. It is a circle around the sun, located at an angle of $180^\circ - 138^\circ = 42^\circ$ relative to the line between you and the sun.

This rainbow is much more difficult to see, in part because of the increased effects of the three factors stated above in the secondary case, but also because you are looking back toward the sun, which essentially drowns out the light from the rainbow. Although I have never been able to seen this tertiary rainbow, it seems quite possible under (highly improbable) ideal conditions, namely, having rain fall generally all around you, except in a path between you and the sun, while at the same time having the sun eclipsed by a properly sized cloud.

Nth-order rainbow: We can now see what happens in general, when the light ray undergoes N reflections inside the raindrop.



With β defined as in the figure, the light gets deflected clockwise by an angle of $180-2\beta$ at each reflection point. In addition, it gets deflected clockwise by an angle of $\arcsin(\frac{4}{3}\sin\beta)-\beta$ at the two refraction points. The total angle of deflection is therefore

$$\Phi = 2\left(\arcsin\left(\frac{4}{3}\sin\beta\right)-\beta\right)+N(180^\circ-2\beta). \tag{17}$$

Taking the derivative to find the extremum gives[4]

$$\sin\beta_0=\sqrt{\frac{9(N+1)^2-16}{16[(N+1)^2-1]}}. \tag{18}$$

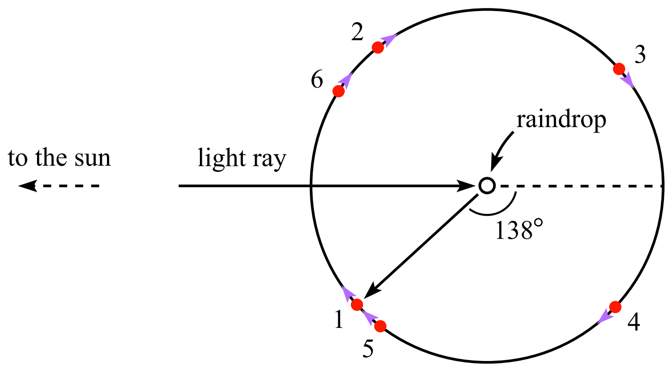
You can check that this agrees with the results in eqs. (4), (11), and (15). Substituting this back into eq. (17) gives the total angle of deflection, Φ, relative to the direction from the sun to you. See the discussion below for an explanation on how Φ relates to the angle ϕ we used above. A few values of Φ are given in the following table.

N	Φ	Φ (mod 360°)
1	138°	138°
2	231°	231°
3	318°	318°
4	404°	44°
5	488°	128°
6	572°	212°

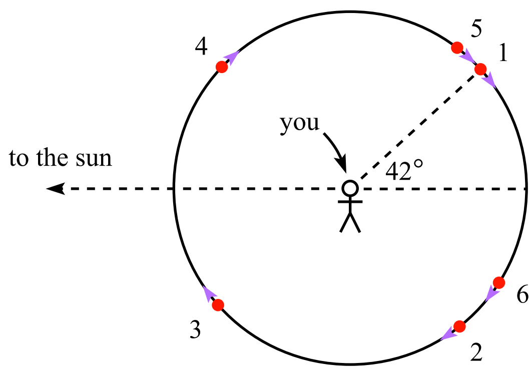
If you take the second derivative of Φ, you will find that it is always positive. Therefore, the extremum is always a minimum. It takes a little thought, though, to deduce the ordering of the colors from this fact. Since the red end of the spectrum has the smallest index of refraction, it is bent the least at the refraction points. Therefore, it has a smaller total angle of deflection than the other colors. The red light will therefore be located (approximately) at the angles given in the above table, while the other colors will occur at larger angles. The locations of the focusing angles

[4]Note that as $N\to\infty$, we have $\sin\beta_0\to 3/4$. Therefore, β approaches the critical angle, 48.6°, which means that the light ray hits the raindrop at nearly 90° with respect to the normal. The cross section of the relevant light rays is therefore very small, which contributes to the faintness of the higher-order rainbows.

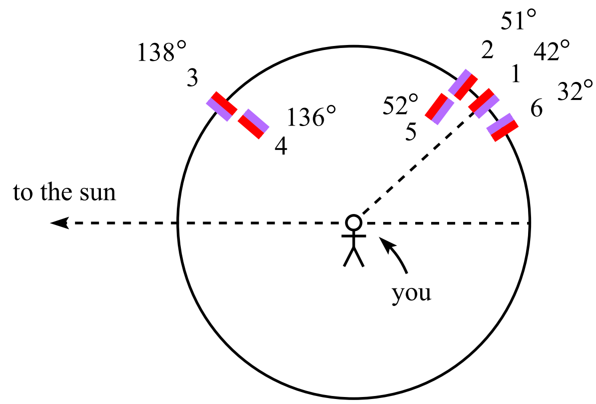
are shown below, where the arrows indicate the direction of the other colors relative to the red. All the arrows point clockwise, because all the extrema are minima.



Since the direction you see the light coming from is the opposite of the direction in which the light travels, you see the light at the following angles. This diagram is obtained by simply rotating the previous one by 180°.



We now note that in the figure preceding eq. (17), we could have had the light ray hitting the bottom part of the raindrop instead of the top, in which case the word "clockwise" would have been replaced with "counterclockwise". This would have the effect of turning Φ into $-\Phi$. The light is, of course, reflected in a whole circle (as long as the ground doesn't get in the way), and this circle is the rainbow that you see; the angles Φ and $-\Phi$ simply represent the intersection of the circle with a vertical plane. We can therefore label each rainbow with an angle between 0 and 180°. This simply means taking the dots and arrows below the horizontal line and reflecting them in the horizontal line. The result is the following figure, which gives the locations and orientations (red on one end, violet on the other) of the first six rainbows.



Note that in the $N \to \infty$ limit, the deflection angle in eq. (17) takes a simple form. In this limit, eq. (18) gives $\beta \approx 48.6^\circ$. Therefore, $\arcsin(\frac{4}{3}\sin\beta) \approx 90^\circ$, and eq. (17) gives

$$\begin{aligned}\Phi &\approx 2\,(90^\circ - 48.6^\circ) + N(180^\circ - 2\cdot 48.6^\circ) \\ &= (N+1)(180^\circ - 2\cdot 48.6^\circ) \\ &= (N+1)82.8^\circ. \end{aligned} \tag{19}$$