Solution

Week 83  (4/12/04)

The brachistochrone

First solution: In the figure below, the boundary conditions are \( y(0) = 0 \) and \( y(x_0) = y_0 \), with downward taken to be the positive \( y \) direction.

From conservation of energy, the speed as a function of \( y \) is \( v = \sqrt{2gy} \). The total time is therefore

\[
T = \int_0^{x_0} \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \, dx. \tag{1}
\]

Our goal is to find the function \( y(x) \) that minimizes this integral, subject to the boundary conditions above. We can therefore apply the results of the variational technique, with a “Lagrangian” equal to

\[
L \propto \frac{\sqrt{1+y'^2}}{\sqrt{y}}. \tag{2}
\]

The Euler-Lagrange equation is

\[
\frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = \frac{\partial L}{\partial y} \implies \frac{d}{dx} \left( \frac{1}{\sqrt{y}} \cdot y' \cdot \frac{1}{\sqrt{1+y'^2}} \right) = -\frac{\sqrt{1+y'^2}}{2y\sqrt{y}}. \tag{3}
\]

Using the product rule on the three factors on the left-hand side, and making copious use of the chain rule, we obtain

\[
-\frac{y'^2}{2y\sqrt{y}\sqrt{1+y'^2}} + \frac{y''}{\sqrt{y}\sqrt{1+y'^2}} - \frac{y'^2y''}{\sqrt{y}(1+y'^2)^{3/2}} = -\frac{\sqrt{1+y'^2}}{2y\sqrt{y}}. \tag{4}
\]

Multiplying through by \( 2y\sqrt{y}(1+y'^2)^{3/2} \) and simplifying gives

\[
-2yy'' = 1+y'^2. \tag{5}
\]

We can integrate this equation if we multiply through by \( y' \) and rearrange to obtain

\[
\int \frac{2yy''}{1+y'^2} = -\int \frac{y'}{y}. \implies \ln(1+y'^2) = -\ln y + A
\]

\[
\implies 1+y'^2 = \frac{B}{y}. \tag{6}
\]
where $B \equiv e^A$. We must now integrate one more time. Solving for $y'$ and separating variables gives
\[
\frac{\sqrt{y} \, dy}{\sqrt{B - y}} = \pm dx. \tag{7}
\]
A helpful change of variables to get rid of the square root in the denominator is $y \equiv B \sin^2 \phi$. Then $dy = 2B \sin \phi \cos \phi \, d\phi$, and eq. (7) simplifies to
\[
2B \sin^2 \phi \, d\phi = \pm dx. \tag{8}
\]
We can now make use of the relation $\sin^2 \phi = (1 - \cos 2\phi)/2$ to integrate this. The result is $B(2\phi - \sin 2\phi) = \pm 2x - C$, where $C$ is an integration constant.

Now note that we may rewrite our definition of $\phi$ (which was $y \equiv B \sin^2 \phi$) as $2y = B(1 - \cos 2\phi)$. If we then define $\theta \equiv 2\phi$, we have
\[
x = \pm a(\theta - \sin \theta) \pm d, \quad y = a(1 - \cos \theta). \tag{9}
\]
where $a \equiv B/2$, and $d \equiv C/2$.

The particle starts at $(x, y) = (0, 0)$. Therefore, $\theta$ starts at $\theta = 0$, since this corresponds to $y = 0$. The starting condition $x = 0$ then implies that $d = 0$. Also, we are assuming that the wire heads down to the right, so we choose the positive sign in the expression for $x$. Therefore, we finally have
\[
x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta). \tag{10}
\]
This is the parametrization of a cycloid, which is the path taken by a point on the rim of a rolling wheel. The initial slope of the $y(x)$ curve is infinite, as you can check.

**Second solution:** Let’s use a variational argument again, but now with $y$ as the independent variable. That is, let the chain be described by the function $x(y)$. The arclength is now given by $ds = \sqrt{1 + x'^2} \, dy$. Therefore, instead of the Lagrangian in eq. (2), we now have
\[
L \propto \frac{\sqrt{1 + x'^2}}{\sqrt{y}}. \tag{11}
\]
The Euler-Lagrange equation is
\[
\frac{d}{dy} \left( \frac{\partial L}{\partial x'} \right) = \frac{\partial L}{\partial x} \quad \implies \quad \frac{d}{dy} \left( \frac{1}{\sqrt{y}} \frac{x'}{\sqrt{1 + x'^2}} \right) = 0. \tag{12}
\]
The zero on the right-hand side makes things nice and easy, because it means that the quantity in parentheses is a constant. Call it $D$. We then have
\[
\frac{1}{\sqrt{y}} \frac{x'}{\sqrt{1 + x'^2}} = D \quad \implies \quad \frac{1}{\sqrt{y}} \frac{dx/dy}{\sqrt{1 + (dx/dy)^2}} = D
\]
\[
\implies \quad \frac{1}{\sqrt{y}} \frac{1}{\sqrt{(dy/dx)^2 + 1}} = D. \tag{13}
\]
This is equivalent to eq. (6), and the solution proceeds as above.
Third solution: Note that the “Lagrangian” in the first solution above, which is given in eq. (2) as
\[ L = \frac{\sqrt{1 + y'^2}}{\sqrt{y}}, \] (14)
is independent of \( x \). Therefore, in analogy with conservation of energy (which arises from a Lagrangian that is independent of \( t \)), the quantity
\[ E \equiv y' \frac{\partial L}{\partial y'} - L = \frac{y'^2}{\sqrt{y} \sqrt{1 + y'^2}} - \frac{\sqrt{1 + y'^2}}{\sqrt{y}} = \frac{-1}{\sqrt{y} \sqrt{1 + y'^2}} \] (15)
is a constant. We have therefore again reproduced eq. (6).