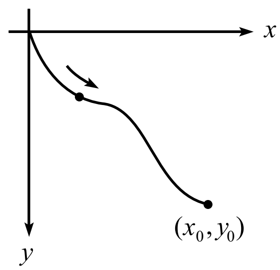
Solution

Week 83 (4/12/04)

The brachistochrone

First solution: In the figure below, the boundary conditions are $y(0) = 0$ and $y(x_0) = y_0$, with downward taken to be the positive y direction.



From conservation of energy, the speed as a function of y is $v = \sqrt{2gy}$. The total time is therefore

$$T = \int_0^{x_0} \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx. \tag{1}$$

Our goal is to find the function $y(x)$ that minimizes this integral, subject to the boundary conditions above. We can therefore apply the results of the variational technique, with a "Lagrangian" equal to

$$L \propto \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,. \tag{2}$$

The Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{1}{\sqrt{y}} \cdot y' \cdot \frac{1}{\sqrt{1+y'^2}}\right) = -\frac{\sqrt{1+y'^2}}{2y\sqrt{y}}\,. \tag{3}$$

Using the product rule on the three factors on the left-hand side, and making copious use of the chain rule, we obtain

$$-\frac{y'^2}{2y\sqrt{y}\sqrt{1+y'^2}} + \frac{y''}{\sqrt{y}\sqrt{1+y'^2}} - \frac{y'^2 y''}{\sqrt{y}(1+y'^2)^{3/2}} = -\frac{\sqrt{1+y'^2}}{2y\sqrt{y}}\,. \tag{4}$$

Multiplying through by $2y\sqrt{y}(1+y'^2)^{3/2}$ and simplifying gives

$$-2yy'' = 1 + y'^2. \tag{5}$$

We can integrate this equation if we multiply through by y' and rearrange to obtain

$$\begin{aligned} \int \frac{2y'y''}{1+y'^2} = -\int \frac{y'}{y}\,. \qquad &\Longrightarrow \qquad \ln(1+y'^2) = -\ln y + A \\ &\Longrightarrow \qquad 1 + y'^2 = \frac{B}{y}\,, \end{aligned} \tag{6}$$

where $B \equiv e^A$. We must now integrate one more time. Solving for y' and separating variables gives

$$\frac{\sqrt{y}\,dy}{\sqrt{B-y}} = \pm\, dx. \tag{7}$$

A helpful change of variables to get rid of the square root in the denominator is $y \equiv B\sin^2\phi$. Then $dy = 2B\sin\phi\cos\phi\; d\phi$, and eq. (7) simplifies to

$$2B\sin^2\phi\; d\phi = \pm\, dx. \tag{8}$$

We can now make use of the relation $\sin^2\phi = (1-\cos 2\phi)/2$ to integrate this. The result is $B(2\phi - \sin 2\phi) = \pm\, 2x - C$, where C is an integration constant.

Now note that we may rewrite our definition of ϕ (which was $y \equiv B\sin^2\phi$) as $2y = B(1-\cos 2\phi)$. If we then define $\theta \equiv 2\phi$, we have

$$x = \pm\, a(\theta - \sin\theta) \pm d, \qquad y = a(1-\cos\theta). \tag{9}$$

where $a \equiv B/2$, and $d \equiv C/2$.

The particle starts at $(x,y) = (0,0)$. Therefore, θ starts at $\theta = 0$, since this corresponds to $y = 0$. The starting condition $x = 0$ then implies that $d = 0$. Also, we are assuming that the wire heads down to the right, so we choose the positive sing in the expression for x. Therefore, we finally have

$$x = a(\theta - \sin\theta), \qquad y = a(1-\cos\theta). \tag{10}$$

This is the parametrization of a *cycloid*, which is the path taken by a point on the rim of a rolling wheel. The initial slope of the $y(x)$ curve is infinite, as you can check.

Second solution: Let's use a variational argument again, but now with y as the independent variable. That is, let the chain be described by the function $x(y)$. The arclength is now given by $ds = \sqrt{1+x'^2}\,dy$. Therefore, instead of the Lagrangian in eq. (2), we now have

$$L \propto \frac{\sqrt{1+x'^2}}{\sqrt{y}}\,. \tag{11}$$

The Euler-Lagrange equation is

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{d}{dy}\left(\frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}}\right) = 0. \tag{12}$$

The zero on the right-hand side makes things nice and easy, because it means that the quantity in parentheses is a constant. Call it D. We then have

$$\begin{aligned} \frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}} = D \qquad &\Longrightarrow \qquad \frac{1}{\sqrt{y}}\frac{dx/dy}{\sqrt{1+(dx/dy)^2}} = D \\ &\Longrightarrow \qquad \frac{1}{\sqrt{y}}\frac{1}{\sqrt{(dy/dx)^2+1}} = D. \end{aligned} \tag{13}$$

This is equivalent to eq. (6), and the solution proceeds as above.

Third solution: Note that the "Lagrangian" in the first solution above, which is given in eq. (2) as

$$L = \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,, \tag{14}$$

is independent of x. Therefore, in analogy with conservation of energy (which arises from a Lagrangian that is independent of t), the quantity

$$E \equiv y'\frac{\partial L}{\partial y'} - L = \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}} = \frac{-1}{\sqrt{y}\sqrt{1+y'^2}} \tag{15}$$

is a constant. We have therefore again reproduced eq. (6).