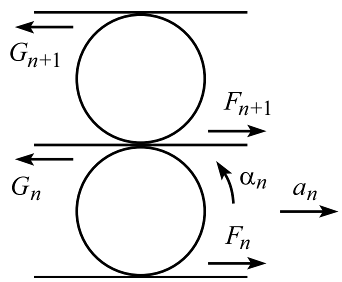
Solution

Week 85 (4/26/04)

Tower of cylinders

Both cylinders in a given row move in the same manner, so we may simply treat them as one cylinder with mass $m = 2M$. Let the forces that the boards exert on the cylinders be labelled as shown. "F" is the force from the plank below a given cylinder, and "G" is the force from the plank above it.



Note that by Newton's third law, we have $F_{n+1} = G_n$, because the planks are massless.

Our strategy will be to solve for the linear and angular accelerations of each cylinder in terms of the accelerations of the cylinder below it. Since we want to solve for two quantities, we will need to produce two equations relating the accelerations of two successive cylinders. One equation will come from a combination of $F = ma$, $\tau = I\alpha$, and Newton's third law. The other will come from the nonslipping condition.

With the positive directions for a and α defined as in the figure, $F = ma$ on the nth cylinder gives

$$F_n - G_n = ma_n, \tag{1}$$

and $\tau = I\alpha$ on the nth cylinder gives

$$(F_n + G_n)R = \frac{1}{2}mR^2\alpha_n \qquad \Longrightarrow \qquad F_n + G_n = \frac{1}{2}mR\alpha_n \tag{2}$$

Solving the previous two equations for F_n and G_n gives

$$\begin{aligned} F_n &= \frac{1}{2}\left(ma_n + \frac{1}{2}mR\alpha_n\right), \\ G_n &= \frac{1}{2}\left(-ma_n + \frac{1}{2}mR\alpha_n\right). \end{aligned} \tag{3}$$

But we know that $F_{n+1} = G_n$. Therefore,

$$a_{n+1} + \frac{1}{2}R\alpha_{n+1} = -a_n + \frac{1}{2}R\alpha_n. \tag{4}$$

We will now use the fact that the cylinders don't slip with respect to the boards. The acceleration of the board above the nth cylinder is $a_n - R\alpha_n$. But the acceleration of this same board, viewed as the board below the $(n+1)$st cylinder, is $a_{n+1} + R\alpha_{n+1}$. Therefore,

$$a_{n+1} + R\alpha_{n+1} = a_n - R\alpha_n. \tag{5}$$

Eqs. (4) and (5) are a system of two equations in the two unknowns, a_{n+1} and α_{n+1}, in terms of a_n and α_n. Solving for a_{n+1} and α_{n+1} gives

$$\begin{aligned} a_{n+1} &= -3a_n + 2R\alpha_n, \\ R\alpha_{n+1} &= 4a_n - 3R\alpha_n. \end{aligned} \tag{6}$$

We can write this in matrix form as

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{7}$$

We therefore have

$$\begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ R\alpha_1 \end{pmatrix}. \tag{8}$$

Consider now the eigenvectors and eigenvalues of the above matrix. The eigenvectors are found via

$$\begin{vmatrix} -3-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} = 0 \qquad \Longrightarrow \qquad \lambda_\pm = -3 \pm 2\sqrt{2}. \tag{9}$$

The eigenvectors are then

$$\begin{aligned} V_+ &= \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_+ = -3 + 2\sqrt{2}, \\ V_- &= \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_- = -3 - 2\sqrt{2}. \end{aligned} \tag{10}$$

Note that $|\lambda_-| > 1$, so $\lambda_-^n \to \infty$ as $n \to \infty$. This means that if the initial $(a_1, R\alpha_1)$ vector has any component in the V_- direction, then the $(a_n, R\alpha_n)$ vectors will head to infinity. This violates conservation of energy. Therefore, the $(a_1, R\alpha_1)$ vector must be proportional to V_+.[1] That is, $R\alpha_1 = \sqrt{2}a_1$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \sqrt{2}a_1 \qquad \Longrightarrow \qquad a_1 = \frac{a}{\sqrt{2}+1} = (\sqrt{2}-1)a. \tag{11}$$

Remark: Let us consider the general case where the cylinders have a moment of inertia of the form $I = \beta MR^2$. Using the above arguments, you can show that eq. (7) becomes

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \frac{1}{1-\beta} \begin{pmatrix} -(1+\beta) & 2\beta \\ 2 & -(1+\beta) \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{12}$$

And you can show that the eigenvectors and eigenvalues are

$$\begin{aligned} V_+ &= \begin{pmatrix} \sqrt{\beta} \\ 1 \end{pmatrix}, \qquad \text{for } \lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}, \\ V_- &= \begin{pmatrix} \sqrt{\beta} \\ -1 \end{pmatrix}, \qquad \text{for } \lambda_- = \frac{\sqrt{\beta}+1}{\sqrt{\beta}-1}. \end{aligned} \tag{13}$$

[1] This then means that the $(a_n, R\alpha_n)$ vectors head to zero as $n \to \infty$, because $|\lambda_+| < 1$. Also, note that the accelerations change sign from one level to the next, because λ_+ is negative.

As above, we cannot have the exponentially growing solution, so we must have only the V_+ solution. We therefore have $R\alpha_1 = a_1/\sqrt{\beta}$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \frac{a_1}{\sqrt{\beta}} \qquad \Longrightarrow \qquad a_1 = \left(\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\right) a. \tag{14}$$

You can verify that all of these results agree with the $\beta = 1/2$ results obtained above.

Let's now consider a few special cases of the

$$\lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1} \tag{15}$$

eigenvalue, which gives the ratio of the accelerations in any level to the ones in the next level down.

- If $\beta = 0$ (all the mass of a cylinder is located at the center), then we have $\lambda_+ = -1$. In other words, the accelerations have the same magnitudes but different signs from one level to the next. The cylinders simply spin in place while their centers remain fixed. The centers are indeed fixed, because $a_1 = 0$, from eq. (14).
- If $\beta = 1$ (all the mass of a cylinder is located on the rim), then we have $\lambda_+ = 0$. In other words, there is no motion above the first level. The lowest cylinder basically rolls on the bottom side of the (stationary) plank right above it. Its acceleration is $a_1 = a/2$, from eq. (14).
- If $\beta \to \infty$ (the cylinders have long massive extensions that extend far out beyond the rim), then we have $\lambda_+ = 1$. In other words, all the levels have equal accelerations. This fact, combined with the $R\alpha_1 = a_1/\sqrt{\beta} \approx 0$ result, shows that there is no rotational motion at any level, and the whole system simply moves to the right as a rigid object with acceleration $a_1 = a$, from eq. (14).