Solution

Week 87 (5/10/04)

Leaving the hemisphere

Assume that the particle slides off to the right. Let v_x and v_y be its horizontal and vertical velocities, with rightward and downward taken to be positive, respectively. Let V_x be the velocity of the hemisphere, with leftward taken to be positive. Conservation of momentum gives

$$mv_x = MV_x \qquad \Longrightarrow \qquad V_x = \left(\frac{m}{M}\right) v_x. \tag{1}$$

Consider the moment when the particle is located at an angle θ down from the top of the hemisphere. Locally, it is essentially on a plane inclined at angle θ, so the three velocity components are related by

$$\frac{v_y}{v_x + V_x} = \tan\theta \qquad \Longrightarrow \qquad v_y = \tan\theta\left(1 + \frac{m}{M}\right) v_x. \tag{2}$$

To see why this is true, look at things in the frame of the hemisphere. In this frame, the particle moves to the right with speed $v_x + V_x$, and downward with speed v_y. Eq. (2) represents the constraint that the particle remains on the hemisphere, which is inclined at an angle θ at the given location.

Let us now apply conservation of energy. In terms of θ, the particle has fallen a distance $R(1-\cos\theta)$, so conservation of energy gives

$$\frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}MV_x^2 = mgR(1-\cos\theta). \tag{3}$$

Using eqs. (1) and (2), we can solve for v_x^2 to obtain

$$v_x^2 = \frac{2gR(1-\cos\theta)}{(1+r)\Big(1+(1+r)\tan^2\theta\Big)}, \qquad \text{where} \quad r \equiv \frac{m}{M}. \tag{4}$$

This function of θ starts at zero for $\theta = 0$ and increases as θ increases. It then achieves a maximum value before heading back down to zero at $\theta = \pi/2$. However, v_x *cannot* actually decrease, because there is no force available to pull the particle to the left. So what happens is that v_x initially increases due to the non-zero normal force that exists while contact remains. But then v_x reaches its maximum, which corresponds to the normal force going to zero and the particle losing contact with the hemisphere. The particle then sails through the air with constant v_x. Our goal, then, is to find the angle θ for which the v_x^2 in eq. (4) is maximum. Setting the derivative equal to zero gives

$$\begin{aligned}
0 &= \Big(1+(1+r)\tan^2\theta\Big)\sin\theta - (1-\cos\theta)(1+r)\frac{2\tan\theta}{\cos^2\theta} \\
\Longrightarrow \qquad 0 &= \Big(1+(1+r)\tan^2\theta\Big)\cos^3\theta - 2(1+r)(1-\cos\theta) \\
\Longrightarrow \qquad 0 &= \cos^3\theta + (1+r)(\cos\theta - \cos^3\theta) - 2(1+r)(1-\cos\theta) \\
\Longrightarrow \qquad 0 &= r\cos^3\theta - 3(1+r)\cos\theta + 2(1+r).
\end{aligned} \tag{5}$$

This is the desired equation that determines θ. It is a cubic equation, so in general it can't be solved so easily for θ. But in the special case of $r = 1$, we have

$$0 = \cos^3\theta - 6\cos\theta + 4. \tag{6}$$

By inspection, $\cos\theta = 2$ is an (unphysical) solution, so we find

$$(\cos\theta - 2)(\cos^2\theta + 2\cos\theta - 2) = 0. \tag{7}$$

The physical root of the quadratic equation is

$$\cos\theta = \sqrt{3} - 1 \approx 0.732 \qquad \Longrightarrow \qquad \theta \approx 42.9^\circ. \tag{8}$$

Alternate solution: In the reference frame of the hemisphere, the horizontal speed of the particle $v_x + V_y = (1 + r)v_x$. The total speed in this frame equals this horizontal speed divided by $\cos\theta$, so

$$v = \frac{(1 + r)v_x}{\cos\theta}\,. \tag{9}$$

The particle leaves the hemisphere when the normal force goes to zero. The radial $F = ma$ equation therefore gives

$$mg\cos\theta = \frac{mv^2}{R}\,. \tag{10}$$

You might be concerned that we have neglected the sideways fictitious force in the accelerating frame of the hemisphere. However, the hemisphere is *not* accelerating beginning at the moment when the particle loses contact, because the normal force has gone to zero. Therefore, eq. (10) looks exactly like it does for the familiar problem involving a fixed hemisphere; the difference in the two problems is in the calculation of v.

Using eqs. (4) and (9) in eq. (10) gives

$$mg\cos\theta = \frac{m(1 + r)^2}{R\cos^2\theta} \cdot \frac{2gR(1 - \cos\theta)}{(1 + r)\Big(1 + (1 + r)\tan^2\theta\Big)}\,. \tag{11}$$

Simplifying this yields

$$\Big(1 + (1 + r)\tan^2\theta\Big)\cos^3\theta = 2(1 + r)(1 - \cos\theta), \tag{12}$$

which is the same as the second line in eq. (5). The solution proceeds as above.

REMARK: Let's look at a few special cases of the $r \equiv m/M$ value. In the limit $r \to 0$ (in other words, the hemisphere is essentially bolted down), eq. (5) gives

$$\cos\theta = 2/3 \qquad \Longrightarrow \qquad \theta \approx 48.2^\circ, \tag{13}$$

a result which may look familiar to you. In the limit $r \to \infty$, eq. (5) reduces to

$$0 = \cos^3\theta - 3\cos\theta + 2 \qquad \Longrightarrow \qquad 0 = (\cos\theta - 1)^2(\cos\theta + 2). \tag{14}$$

Therefore, $\theta = 0$. In other words, the hemisphere immediately gets squeezed out very fast to the left.

For other values of r, we can solve eq. (5) either by using the formula for the roots of a cubic equation (very messy), or by simply doing things numerically. A few numerical results are:

r	$\cos\theta$	θ
0	.667	48.2°
1/2	.706	45.1°
1	.732	42.9°
2	.767	39.9°
10	.858	30.9°
100	.947	18.8°
1000	.982	10.8°
∞	1	0°