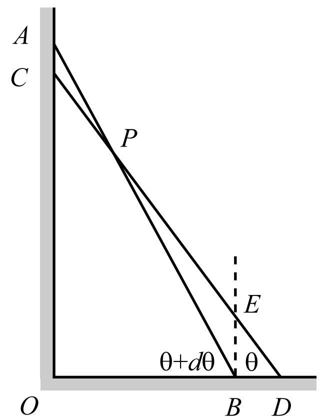
Solution

Week 88 (5/17/04)

Ladder envelope

Let the ladder have length 1, for simplicity. In the figure below, let the ladder slide from segment AB to segment CD. Let CD make an angle θ with the floor, and let AB make an angle $\theta + d\theta$, with $d\theta$ very small. The given problem is then equivalent to finding the locus of intersections, P, of adjacent ladder positions AB and CD.



Put the ladder on a coordinate system with the floor as the x-axis and the wall as the y-axis. Let a vertical line through B intersect CD at point E. We will find the x and y coordinates of point P by determining the ratio of similar triangles ACP and BEP. We will find this ratio by determining the ratio of AC to BE. AC is given by

$$AC = \sin(\theta + d\theta) - \sin\theta \approx \cos\theta\, d\theta, \tag{1}$$

which is simply the derivative of $\sin\theta$ times $d\theta$. Similarly,

$$BD = \cos\theta - \cos(\theta + d\theta) \approx \sin\theta\, d\theta, \tag{2}$$

which is simply the negative of the derivative of $\cos\theta$ times $d\theta$. BE is then given by

$$BE = BD\tan\theta \approx \tan\theta\sin\theta\, d\theta. \tag{3}$$

The ratio of triangle ACP to triangle BEP is therefore

$$\frac{\triangle ACP}{\triangle BEP} = \frac{AC}{BE} \approx \frac{\cos\theta}{\tan\theta\sin\theta} = \frac{\cos^2\theta}{\sin^2\theta} \equiv r. \tag{4}$$

The x coordinate of P is then, using $OB = \cos(\theta + d\theta) \approx \cos\theta$,

$$P_x = \frac{r}{1+r}(OB) \approx \frac{\cos^2\theta}{\sin^2\theta + \cos^2\theta}(OB) \approx \cos^3\theta. \tag{5}$$

Likewise, the y coordinate of P is $P_y = \sin^3\theta$. The envelope of the ladder may therefore be described parametrically by

$$(x, y) = (\cos^3\theta, \sin^3\theta), \qquad \pi/2 \ge \theta \ge 0. \tag{6}$$

Equivalently, using $\cos^2\theta + \sin^2\theta = 1$, the envelope may be described by the equation,

$$x^{2/3} + y^{2/3} = 1. \tag{7}$$