Solution

Week 89 (5/24/04)

Rope between inclines

Let the total mass of the rope be m, and let a fraction f of it hang in the air. Consider the right half of this section. Its weight, $(f/2)mg$, must be balanced by the vertical component, $T\sin\theta$, of the tension at the point where it joins the part of the rope touching the right platform. The tension at this point therefore equals $T = (f/2)mg/\sin\theta$.

Now consider the part of the rope touching the right platform. This part has mass $(1-f)m/2$. The normal force from the platform is $N = (1-f)(mg/2)\cos\theta$, so the maximal friction force equals $(1-f)(mg/2)\cos\theta$, because $\mu = 1$. This fiction force must balance the sum of the gravitational force component along the plane, which is $(1-f)(mg/2)\sin\theta$, plus the tension at the lower end, which is the $(f/2)mg/\sin\theta$ we found above. Therefore,

$$\frac{1}{2}(1-f)mg\cos\theta = \frac{1}{2}(1-f)mg\sin\theta + \frac{fmg}{2\sin\theta}\,, \tag{1}$$

which gives

$$f = \frac{F(\theta)}{1+F(\theta)}\,, \qquad \text{where} \quad F(\theta) \equiv \cos\theta\sin\theta - \sin^2\theta. \tag{2}$$

This expression for f is a monotonically increasing function of $F(\theta)$, as you can check. The maximal f is therefore obtained when $F(\theta)$ is as large as possible. Using the double-angle formulas, we can rewrite $F(\theta)$ as

$$F(\theta) = \frac{1}{2}(\sin 2\theta + \cos 2\theta - 1). \tag{3}$$

The derivative of this is $\cos 2\theta - \sin 2\theta$, which equals zero when $\tan 2\theta = 1$. Therefore,

$$\theta_{\text{max}} = 22.5^\circ. \tag{4}$$

Eq. (3) then yields $F(\theta_{\text{max}}) = (\sqrt{2}-1)/2$, and so eq. (2) gives

$$f_{\text{max}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2} \approx 0.172. \tag{5}$$