

Solution

Week 90 (5/31/04)

The game of NIM

As with many problems, this one can be solved in two possible ways. We can (1) write down the correct answer, through some stroke of genius, and then verify that it works, or (2) work out some simple cases, get a feel for the problem, and eventually wind our way around to the correct answer. For the problem at hand, let's proceed via the second method and try to arrive at the answer with some motivation.

We'll start by working out what happens in particular cases of small numbers of coins in the piles, and then we'll look for a pattern. A reasonable way to organize the results is to determine which combinations of numbers are guaranteed losing positions (assuming that both players are aware of the optimal strategy).

The most obvious losing position (LP) is piles with coins of numbers $(1,1,0)$. If you encounter this setup, you must pick one coin, and then your opponent will pick the last one and thereby win. More generally, the combination $(N, N, 0)$ is an LP, because if you take off n coins from one pile, your opponent will take off n coins from the other. He will keep matching you on each turn, until finally the situation is $(0,0,0)$, with his having removed the last coin(s). Note that situations of $(N, M, 0)$ and (N, N, M) are therefore winning positions (WP), because it is possible to turn them into an LP with one move.

This reasoning utilizes the following two obvious properties that an LP must have: (1) removal of any number of coins from one pile of an LP creates a non-LP, and (2) on the next turn it is always possible to bring the situation back to an LP again.

Consider now the cases where no two piles have the same numbers of coins. The smallest case is $(1,2,x)$. $x = 3$ is the first possibility for an LP, and we quickly see that it is indeed an LP, because the removal of any number of coins from any of the piles yields a setup of the form $(N, M, 0)$ or (N, N, M) , which are WP's, as we saw above.

Note that once we have found an LP, we know that any triplet that has two numbers in common with the LP, with its remaining number larger than that in the LP, must be a WP. This is true because it is possible to turn it into an LP by removing coins from this last pile.

If we look at other triplets in which the first number is 1, we find that $(1,4,5)$, $(1,6,7)$, $(1,8,9)$, etc., are LP's, as you can check by showing that any move turns them into a WP.

Now consider cases where 2 is the smallest number of coins in a pile. We find, after a little fiddling, that $(2,4,6)$, $(2,5,7)$, $(2,8,10)$, and $(2,9,11)$, etc., are LP's, as you can check.

Similar fiddling, starting with a 3, gives $(3,4,7)$, $(3,5,6)$, $(3,8,11)$, and $(3,9,10)$, etc., as LP's.

Let's now make a table of this hodgepodge of results, for up to seven coins in a pile. The two axes will be the first two numbers in an LP triplet, and the entry in the table will be the third. The table is of course symmetric.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

As a first guess at the key to this table, we might say that two numbers in an LP triplet must add up to the third. This, however, does not work for the (3,5,6) triplet. It also does not work for the (3,9,10) triplet that we found above. Continuing on to higher numbers, the guess seems to work for triplets starting with a 4. But then if we start with a 5, we eventually find the LP triplets (5,9,12) and (5,11,14), for which the sum of two numbers doesn't equal the third.

In an effort to find the key, let us exploit the patterns in the table, perhaps brought out best by the following grouping:

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

The entries in the upper right 4×4 box are 4 more than the corresponding entries in the upper left box. Likewise, within each 4×4 box, the entries in the upper right 2×2 box are 2 more than the entries in the upper left box. Similar results would be evident if we doubled the size of the box (out to 15), where we would see 8×8 boxes having entries differing by 8.

All this suggests that powers of 2 are important in this problem. We therefore should consider writing the numbers in a way where factors of 2 are evident, that is, in base 2. There is no guarantee that this will help, but let's try it and see what happens. Let's write down the troublesome triplets we've found (the ones for which two of the numbers don't add up to the third) in base 2:

3:	11	3:	11	5:	101	5:	101
5:	101	9:	1001	9:	1001	11:	1011
6:	110	10:	1010	12:	1100	14:	1110

What property do these triplets have? When written in the above form, we see that each column in base 2 contains an even number of 1's. After checking some

other triplets, this appears to be true in general for an LP. We will prove this with the following theorem.

Theorem: *Call a triplet an E-triplet (the “E” stands for “even”) if it has the following property: When the three numbers are written in base 2, there is an even number (that is, either zero or two) of 1’s in each digit’s place. Then a triplet is a losing position (LP) if and only if it is an E-triplet.*

Proof: Let us establish the following three facts concerning E-triplets:

1. Removal of any number of coins from one pile of an E-triplet will turn the triplet into a non-E-triplet.
2. Given a non-E-triplet, it is always possible to remove coins from one pile to turn the triplet into an E-triplet.
3. (0,0,0) is an E-triplet.

These facts may be demonstrated as follows:

1. This is true because any two numbers in an E-triplet uniquely determine the third.
2. We will describe how to turn any non-E-triplet into an E-triplet. Write the three numbers of coins in base 2, and put them in a column, with the unit’s digits aligned, as we did above. Starting from the *left*, look at each digit’s column until you find a column with an odd number (that is, either one or three) of 1’s. Let this be the n th column (counting from the *right*).

If there is one 1 in the n th column, label the number containing this 1 as A . If there are three 1’s, then arbitrarily pick any of the numbers to be A . Remove coins from A by switching the 1 in the n th column to a 0, and also by switching any 1’s to 0’s, or 0’s to 1’s, in other columns to the right of the n th column, in order to produce an even number of 1’s in all columns. We have now created an E-triplet.

Note that this switching of 1’s and 0’s does indeed correspond to removing (as opposed to adding) coins from A , because even if all the columns to the right of the n th column involve switching 0’s to 1’s, this addition of $2^{n-1} - 1$ coins is still less than the subtraction of the 2^{n-1} coins arising from the 1-to-0 switch in the n th column.

3. This is true, by definition of an E-triplet.

The first two of these facts show that if player X receives an E-triplet on a given turn, then player Y can ensure that X receives an E-triplet on every subsequent turn. Therefore, X must always *create* a non-E-triplet, by the first of the three facts. X therefore cannot take the last coin, because he cannot create the E-triplet (0, 0, 0). Therefore, an E-triplet is a losing position. ■

The best strategy in this game is therefore to give your opponent an E-triplet whenever you can. If both players are aware of this strategy, then the outcome of

the game is determined by the initial piles of coins. If they form an E-triplet, then the player who goes first loses. If they do not form an E-triplet, then the player who goes first wins, because he can always create an E-triplet to give to his opponent.

REMARKS: If the starting numbers of coins are random, then the player who goes first will most likely win, because most triplets are not E-triplets. We can demonstrate this fact by using the somewhat crude scenario where the three numbers are random numbers from 0 to $2^n - 1$, that is, they each have n digits in base 2 (many of which may be zero). Then there are $(2^n)^3$ possible triplets. But there are only 4^n possible E-triplets, because each of the n columns of three digits (when we write the three numbers on top of each other) has four E-triplet possibilities: 0,0,0; 1,1,0; 1,0,1; and 0,1,1. The fraction of E-triplets is therefore $4^n/2^{3n} = 2^{-n}$, which goes to zero for large n .

Note that there is nothing special about having three piles. We can have any number of piles (but still two players), and all the above reasoning still holds. Losing positions are the ones that have an even number of 1's in each column, when written in base 2. The three facts in the above theorem still hold.